Qubit 2003

Quantum Computation Without Entanglement

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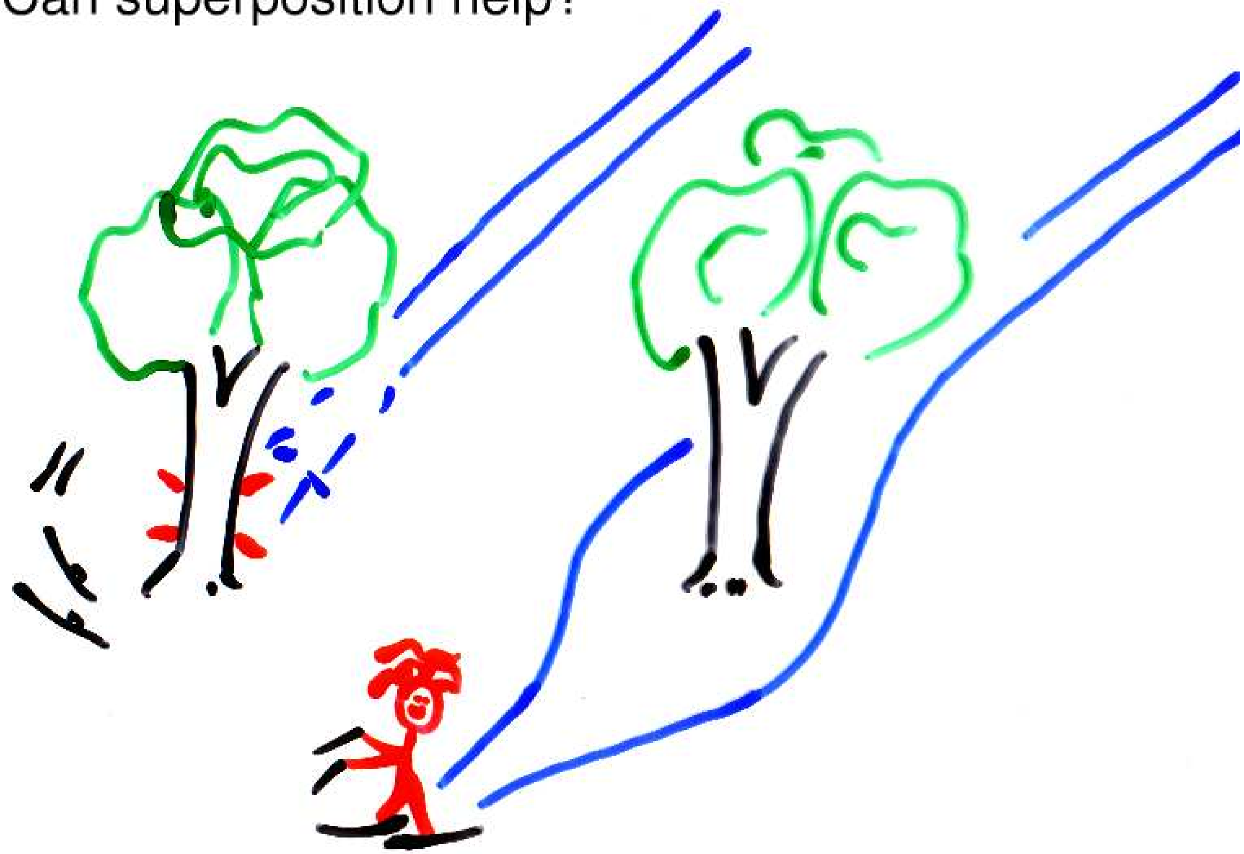
Superposition Can Be Useful

Can superposition help?



Is Entanglement Necessary?

"For any quantum algorithm operating on pure states we prove that the presence of multi-partite entanglement [. . .] is necessary if the quantum algorithm is to offer an exponential speed-up over classical computation."

— Jozsa and Linden, quant-ph/0201143

Separable Pure States

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Proof: Separable two-qubit pure states can be written as

$$\begin{aligned} & (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle) \\ = \quad & \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle \, . \end{aligned}$$

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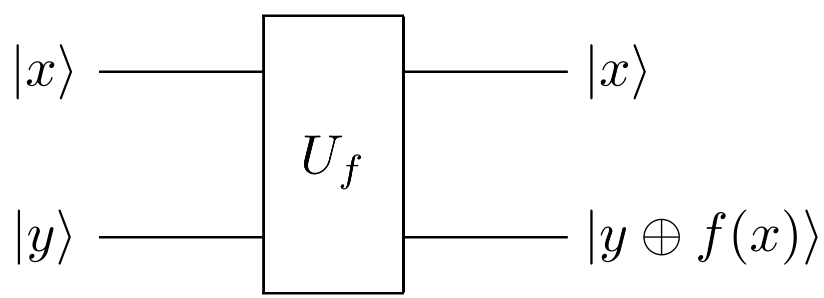
No choice of $\alpha, \beta, \gamma, \delta$ can induce $\alpha\gamma = \beta\delta = \frac{1}{\sqrt{2}}$ and $\alpha\delta = \beta\gamma = 0$ because the first equation requires that $\alpha\gamma\beta\delta = \frac{1}{2}$ and the second requires that $\alpha\delta\beta\gamma = 0$.

Quantum Computation

Consider function $f : \{0,1\}^n \rightarrow \{0,1\}^m$.



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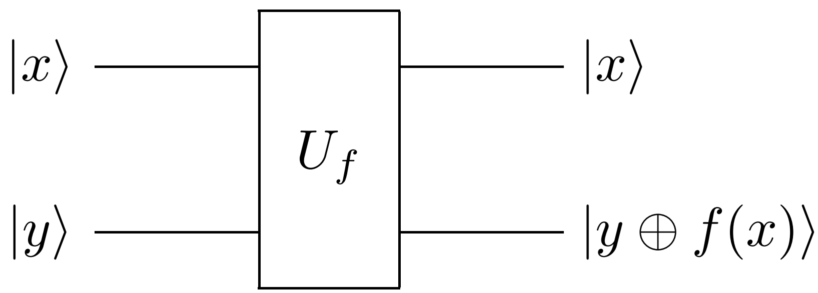

Exponentially many values can be computed simultaneously if we start with a superposition.

$$U_f \sum_{i=1}^{2^n} \alpha_i |x_i\rangle |y\rangle = \sum_{i=1}^{2^n} \alpha_i |x_i\rangle |y \oplus f(x_i)\rangle$$

Deutsch's Problem

Consider function $f : \{0, 1\} \rightarrow \{0, 1\}$.

We want to know whether or not $f(0) = f(1)$.

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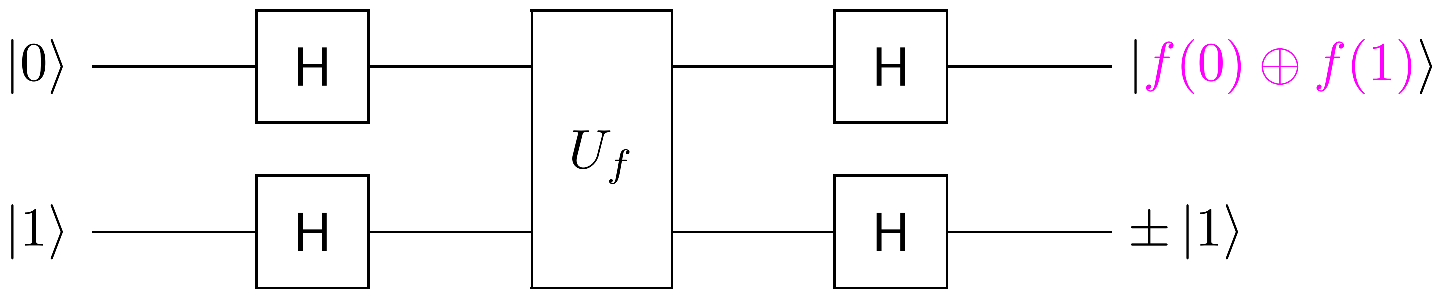

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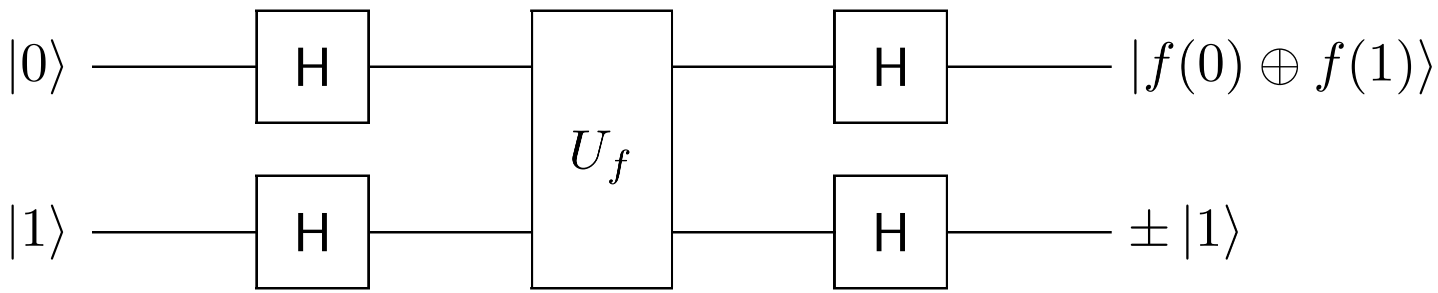

We can determine whether or not $f(0) = f(1)$ with a single call on a circuit that computes function f.

This would be impossible for a classical computer!

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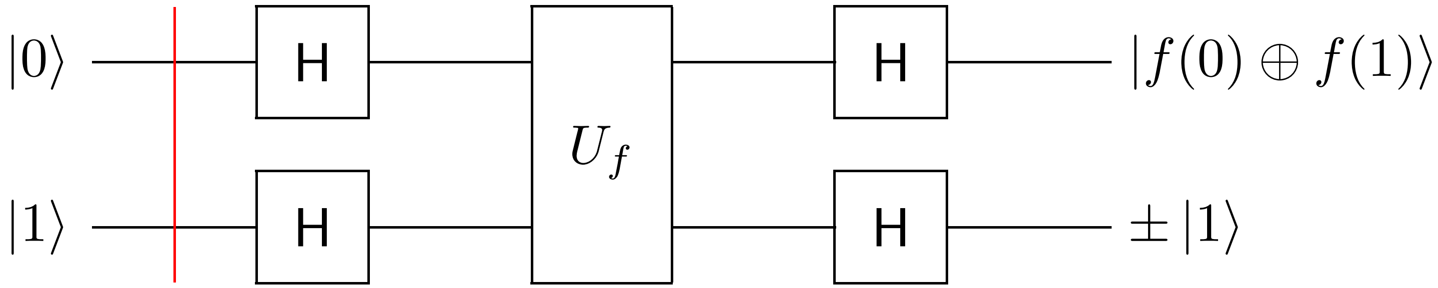

No entanglement here.

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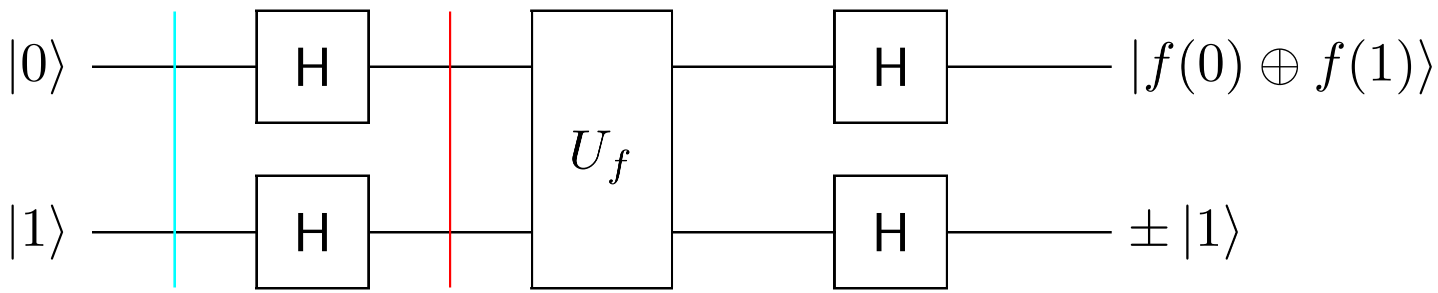

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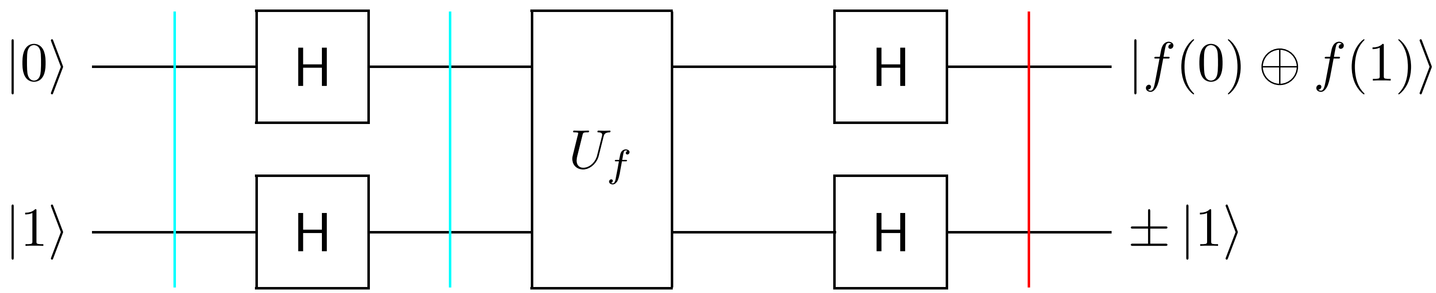

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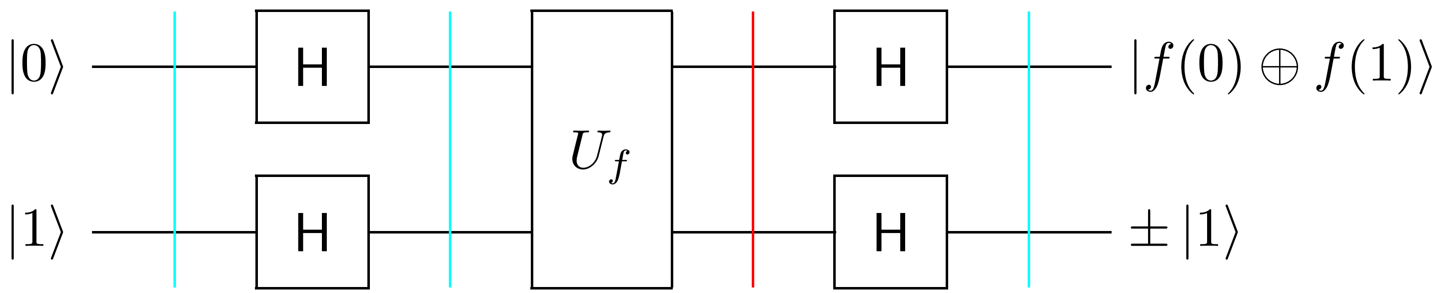

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No entanglement anywhere!

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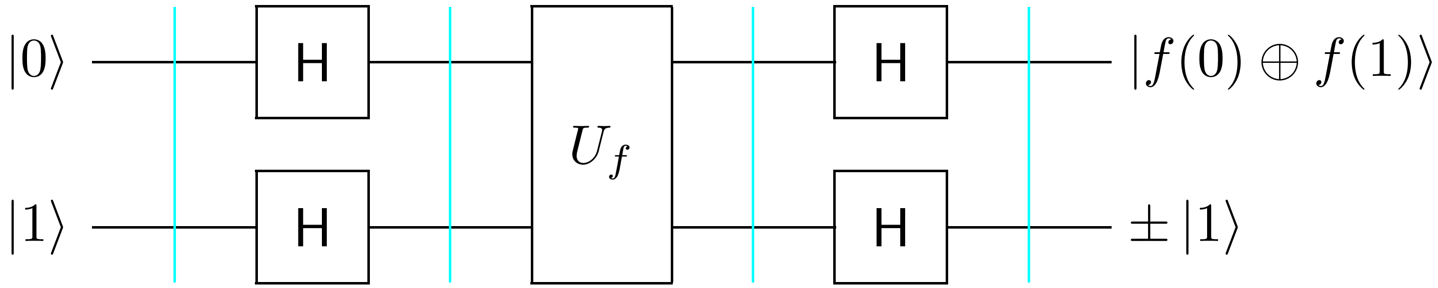

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Really?

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Consider functions $f_0 : \{0,1\}^n \rightarrow \{0,1\}$ and $f_1 : \{0,1\}^n \rightarrow \{0,1\}$.
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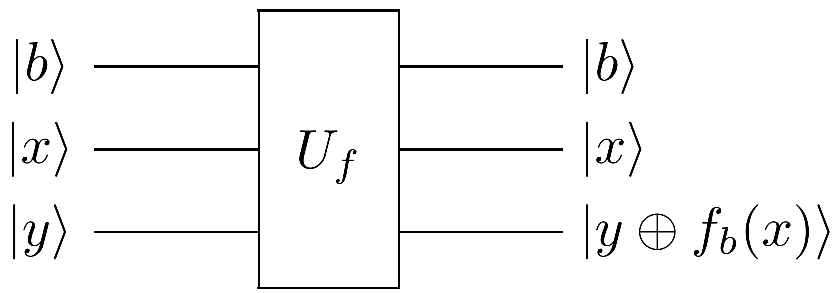

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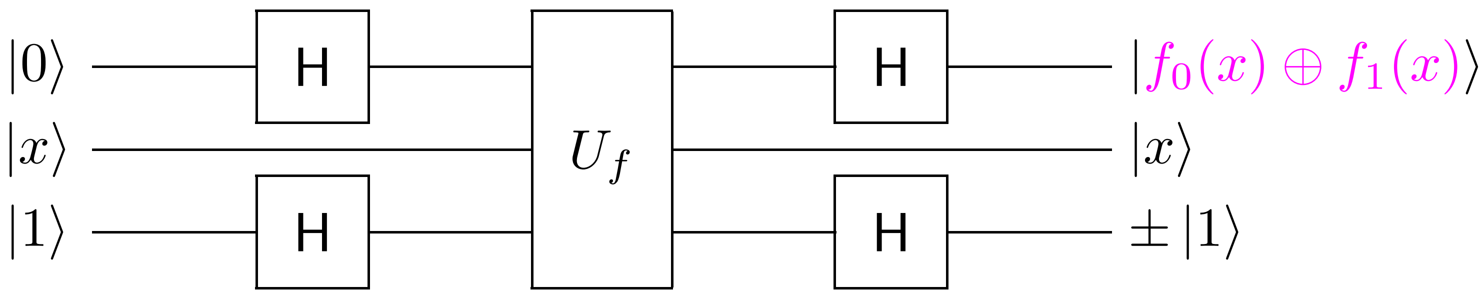

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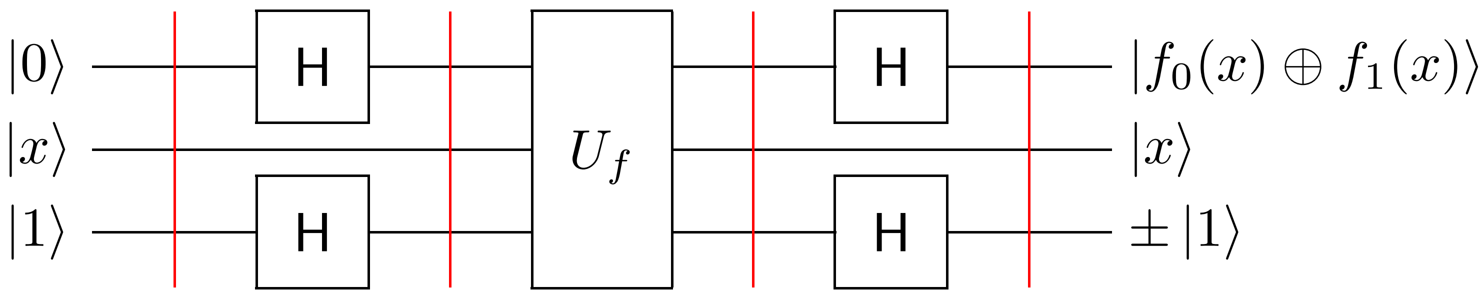

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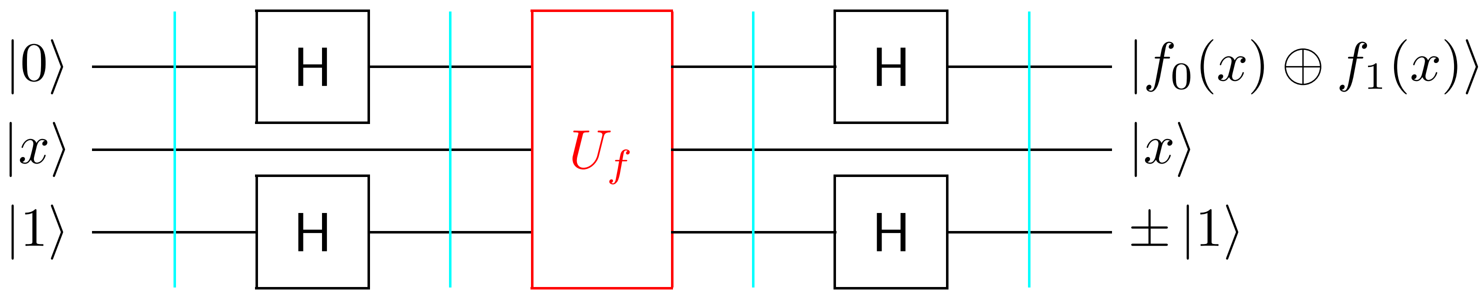

No entanglement here

Lots of entanglement there!

Mixed States

Definition: A mixed state ρ is *separable* if it can be written as

$$\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$$

where each $|\psi_i\rangle = |\psi_i\rangle_A \otimes |\psi_i\rangle_B$ is separable.

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Such states can be prepared by local operations at A and B given classical communication *and the power of forgetting*:

- ◇ A chooses some i with probability p_i and tells B the choice of i;
- ◇ A prepares $|\psi_i\rangle_A$ and B prepares $|\psi_i\rangle_B$; now they share $|\psi_i\rangle$;
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This is very different from requiring that $\rho = \rho_A \otimes \rho_B$.

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Both

$$\rho_{\pm} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \pm 1 & 0 \\ 0 & \pm 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

are entangled.

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are entangled, yet their equal mixture

$$\frac{1}{2}\rho_+ + \frac{1}{2}\rho_- \;=\; \frac{1}{2}|01\rangle\langle 01| + \frac{1}{2}|10\rangle\langle 10|$$

is separable.

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Pseudo-purity ε is conserved by unitary operations.

Separable Pseudo-Pure States

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In any dimension N and for any $|\psi\rangle$, the pseudo-pure state

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These pseudo-pure states appear naturally in NMR experiments.

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- "Whether or not entanglement is a necessary condition for quantum computation is a question of fundamental importance", Linden & Popescu, PRL 87(4)047901, 2001.
- "Can this [using small ε] provide a computational benefit (over classical computations) in the total absence of entanglement?", Jozsa & Linden, quant-ph/0201143, 2002.

The Deutsch-Jozsa Problem

Function $f : \{0,1\}^n \to \{0,1\}$ is promised to be *constant* or *balanced*.

DJ's problem: Decide which is the case.

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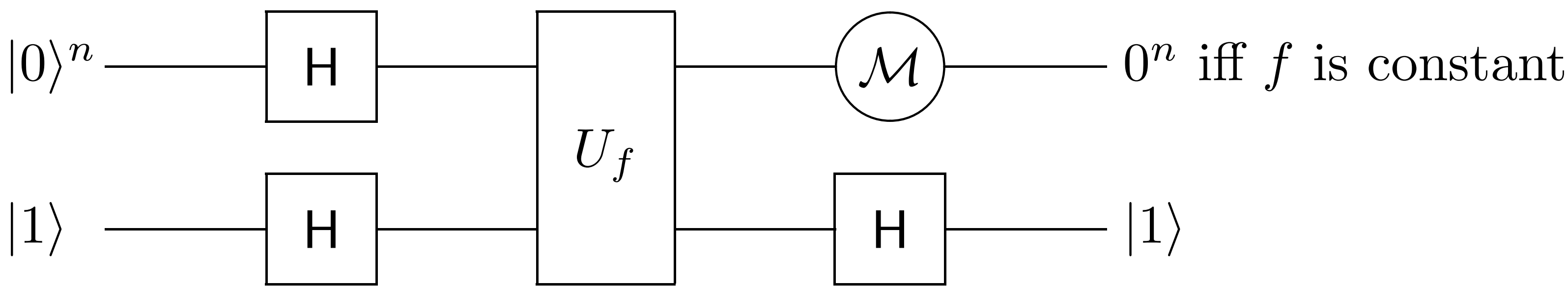

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We consider the following three cases.

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We demonstrate the power of quantum computation without entanglement by showing cases in which more information can be obtained in the third case than in the first.

DJ — Information Gained by One Query

Assume *a priori* that f is balanced with probability $\frac{1}{2}$ and constant with probability $\frac{1}{2}$. The amount of information we *lack* about which is the case is exactly one bit.

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How much of this information I can be gained by a single function evaluation?

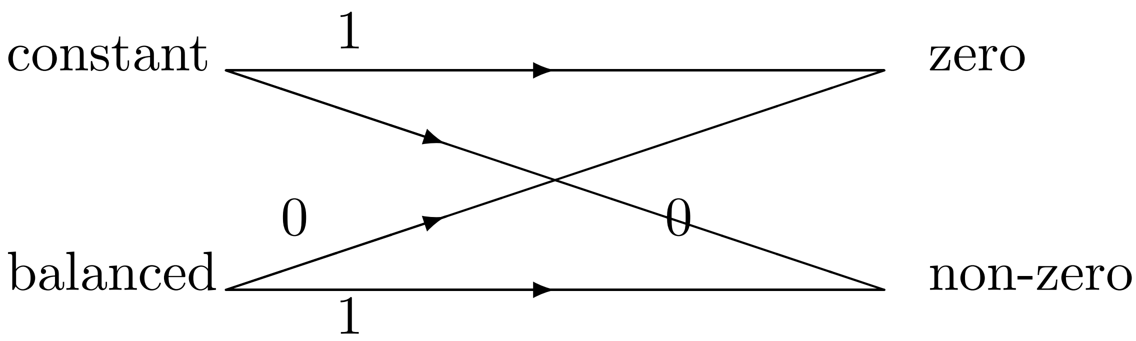
Classical Computation

Nothing is gained. $I = 0$.

Whatever x we choose, a single value of $f(x)$ tells us nothing about whether the function is balanced or constant.

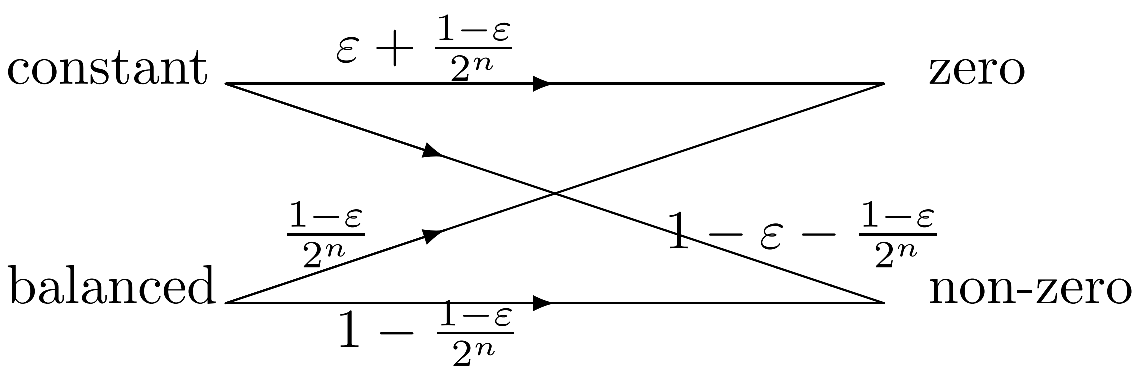
Pure Quantum Computation

Complete knowledge is obtained after a single query. $I = 1$.



Quantum Computation Without Entanglement

If we apply the Deutsch-Jozsa algorithm on a pseudo-pure state, instead of the pure state $|0\rangle^n|1\rangle$, we do obtain *some* information.



Even if $\varepsilon < \frac{2}{N^2}$ is below the Braunstein, Caves, Jozsa, Linden, Popescu and Schack bound.

$$I = h(p) - p_0 h\left(\frac{p}{p_0}\left(\varepsilon + \frac{1-\varepsilon}{2^n}\right)\right) + (1-p_0)h\left(\frac{p(1-\varepsilon)}{1-p_0}\left(1 - \frac{1}{2^n}\right)\right) > 0$$

where

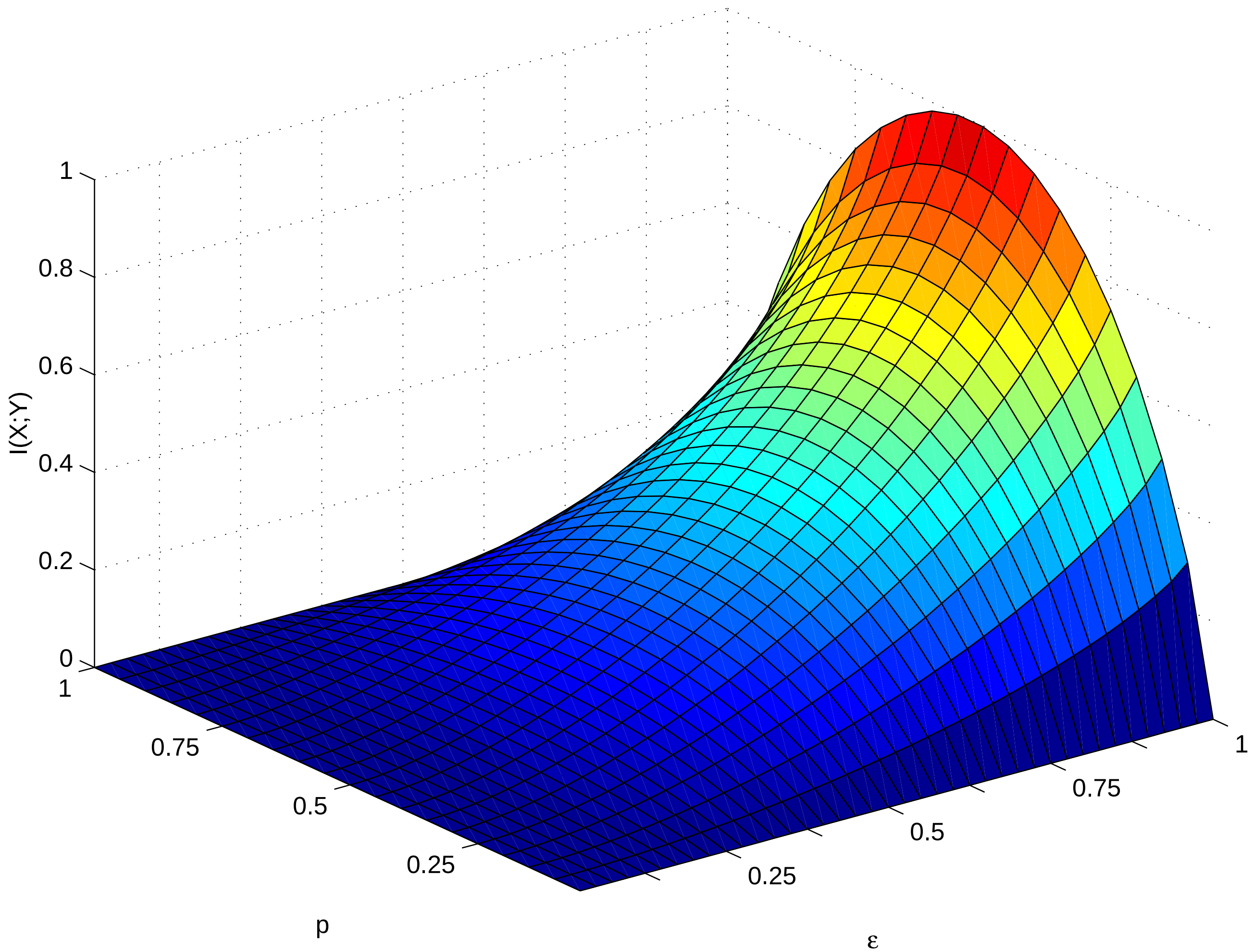
$$p_0 = \frac{1-\varepsilon}{2^n} + \varepsilon p$$

and

$$h(q) \equiv -q\log_2 q - (1-q)\log_2(1-q)$$

is the Shannon binary entropy function.

DJ — Information Gained by One Query



Simon's Problem

Consider two-to-one function $f : \{0,1\}^n \to \{0,1\}^{n-1}$.
There is a single nonzero s such that $f(x) = f(x \oplus s)$ for all x.
Simon's problem: find s.

- ◇ Classical solution: $\Theta(2^{n/2})$ queries are necessary and sufficient (by the birthday "paradox").
- ◇ Quantum solution: $\Theta(n)$ queries in the expected sense with Simon's original algorithm.
- ◇ Exact quantum solution: $\Theta(n)$ queries in the worst case [BH97].

Simon — Information Gained by One Query

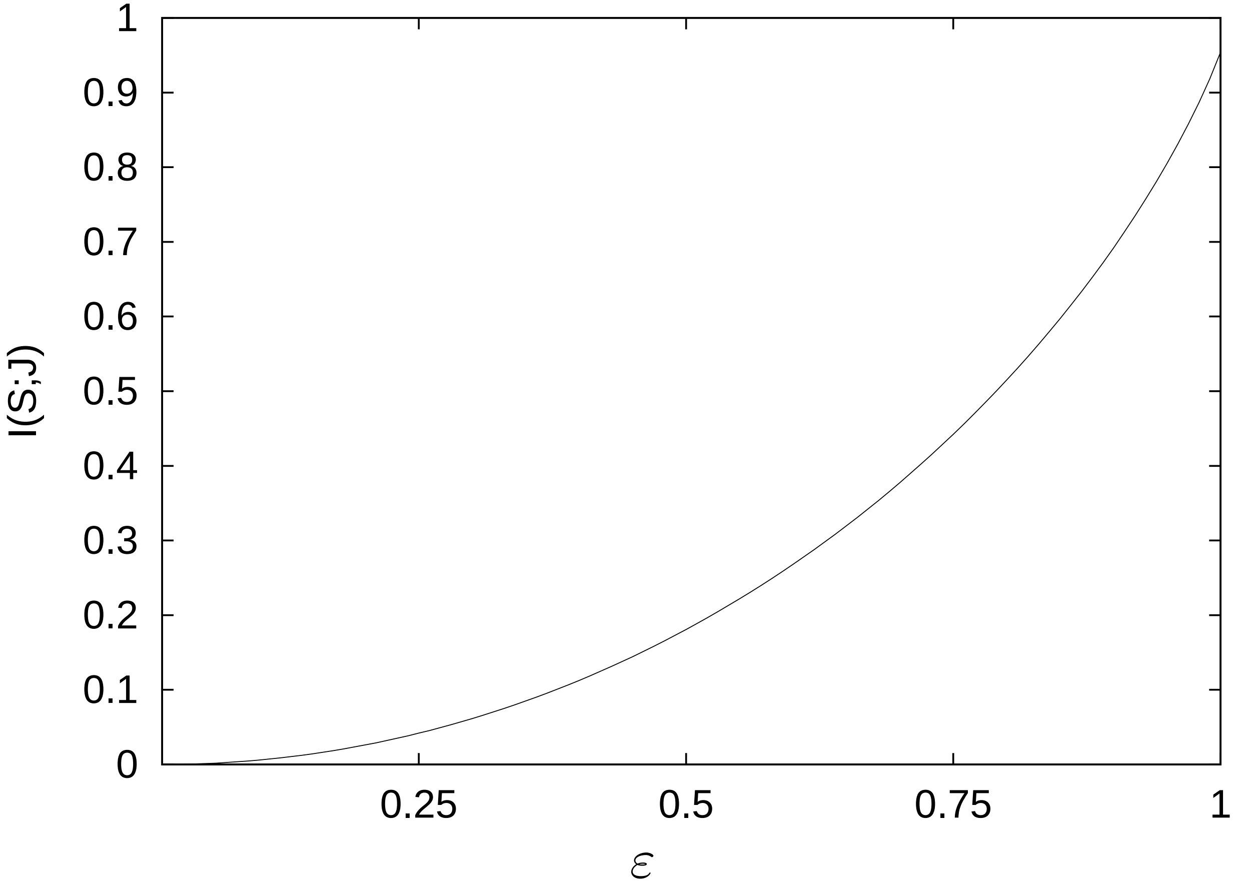
Assume s is selected uniformly from $\{1...2^n - 1\}$. The amount of information we lack about its value is $\log(2^n - 1) \approx n - O(2^{-n})$.

How much of this information can be obtained using one query?

- If it's classical query—nothing.
- If it's the first *quantum* query of Simon's algorithm—almost one bit.
- And with pseudo-pure state, it is

$$\left(2^{n-1} - 1\right) \frac{1+\varepsilon}{2^n} \log \frac{1+\varepsilon}{2^n}$$
$$- \left(1 - \frac{1+\varepsilon}{2^n}\right) \log \frac{1 - \frac{1+\varepsilon}{2^n}}{2^n - 1}$$
$$+ \frac{1-\varepsilon}{2} \log \left(\frac{1-\varepsilon}{2^n}\right) > 0$$

Simon — Information Gained by One Query



Conclusions

- Quantum computing without entanglement *is* possible.
- There is potential evidence that *bound entanglement* is sufficient for making Grover search better than classical (using more than one query).

Limits

- The advantage we found is tiny—exponentially small.
- Entanglement is still required for all practical purposes!
  (so far)

Open Questions

◇ Find cases for which quantum computing without entanglement provides a non-negligible advantage over classical computation.

◇ Find examples in which the Quantum Computation Without Entanglement advantage persists for more than one query.

◇ What does this *really* tell us about why quantum computers (may) have a computational advantage over classical computers?

◇ What does this *really* tell us about how separability is a richer notion for mixed states compared to pure states?

FIN