Sufficient conditions for a disentanglement

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We consider a disentanglement process in which local properties of an entangled state are preserved, while the entanglement between the subsystems is erased. Sufficient conditions for a perfect disentanglement (into product states and into separable states) are derived, and connections to the conditions for perfect cloning and for perfect broadcasting are observed. [S1050-2947(99)00712-X]

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Since the introduction of the paradox of Einstein, Podolsky, and Rosen [1] and Schrödinger's cat [2], the entanglement of quantum states, i.e., the ability of composite quantum systems to exhibit nonlocal correlations, has been one of the most fascinating consequences of the quantum formalism. Today the entanglement also serves as a basic resource of quantum information theory [3], which is based upon the idea that the carriers of information are physical objects and as such are subjected to the laws of quantum mechanics.

Being of such an importance both for the foundations of quantum mechanics and in the quantum information theory, the phenomenon of entanglement has been extensively studied. Recent studies include characterization and classification of the entanglement, manipulation of the entanglement, and its potential applications [4]. The interplay of the local and global properties of the composite systems is both interesting and important in this context. One of the questions that can be asked is the possibility of erasing the entanglement between the subsystems, while keeping their local properties intact [5].

Two different kinds of the *disentanglement* procedure were suggested [5]: disentanglement to the product state of the reduced density matrices,

$$\rho \to \rho_{dis} = \mathrm{tr}_B \rho \otimes \mathrm{tr}_A \rho, \tag{1}$$

and disentanglement into separable states,

$$\rho \to \rho_{dis} = \sum_i w_i \rho_i^A \otimes \rho_i^B, \tag{2}$$

with ρ_{dis} being a state that satisfies $\mathrm{tr}_A \rho = \mathrm{tr}_A \rho_{dis}$, $\mathrm{tr}_B \rho = \mathrm{tr}_B \rho_{dis}$, thus, a state that has the same local properties as the original state ρ. As it was shown by Terno [5] and Mor [6] an unknown entangled state cannot be disentangled neither to product, nor to separable states.

Another question is the possibility of a state-dependent disentanglement. By this we understand a procedure that operates on a secretly chosen state, ρ_i, which belongs to some predefined set of entangled states $\{\rho^1, \ldots, \rho^n\}$. The output of this process is a corresponding state ρ^i_{dis}.

Several sets were analyzed as possible inputs of a state-dependent disentanglement machine. One of the sets that cannot be disentangled into product states consists of the two possible outputs of the state-dependent cloning machine [7], which operates obliviously on two arbitrary pure states ψ_1 and ψ_2. Their optimal copies cannot be disentangled into product states, since it would increase their distinguishabiltity beyond its optimal value [5].

The states

$$|\psi_0\rangle = c_\phi \begin{pmatrix} c_\theta \\ s_\theta \end{pmatrix} \begin{pmatrix} c_\theta \\ s_\theta \end{pmatrix} + s_\phi \begin{pmatrix} s_\theta \\ -c_\theta \end{pmatrix} \begin{pmatrix} s_\theta \\ -c_\theta \end{pmatrix},$$

$$|\psi_1\rangle = c_\phi \begin{pmatrix} c_\theta \\ -s_\theta \end{pmatrix} \begin{pmatrix} c_\theta \\ -s_\theta \end{pmatrix} + s_\phi \begin{pmatrix} s_\theta \\ c_\theta \end{pmatrix} \begin{pmatrix} s_\theta \\ c_\theta \end{pmatrix}, \tag{3}$$

with $c_\phi \equiv \cos \phi$, etc., also cannot be disentangled into product states [5,6]. On the other hand, pairs of maximally entangled states, not necessary orthogonal, can be disentangled [6]:

The following set of states [6]

$$|\psi_0\rangle = |00\rangle,$$

$$|\psi_1\rangle = |11\rangle,$$

$$|\psi_2\rangle = (1/\sqrt{2})|00\rangle + |11\rangle, \tag{4}$$

can be disentangled into a mixture of tensor product states; but not into product states, and finally, the following set of states [6]:

$$|\psi_0\rangle = |00\rangle,$$

$$|\psi_1\rangle = |11\rangle,$$

$$|\psi_2\rangle = (1/\sqrt{2})[|00\rangle + |11\rangle],$$

$$|\psi_3\rangle = (1/\sqrt{2})|++\rangle, \tag{5}$$

where $|+\rangle = (1/\sqrt{2})(|0\rangle + |1\rangle)$, cannot be disentangled at all. An approximate disentanglement [8] and disentanlement

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restricted to local processes [9] were recently also investigated. It should be noted that if two copies of the system are available, any state can be trivially disentangled. Namely, we trace out the first subsystem of the first copy and the second one of the secondy copy. Thus we are left with $\rho_{dis} = \mathrm{tr}_B\rho \times \mathrm{tr}_A\rho$.

In this paper we present sufficient conditions for the exact disentanglement. The conditions for disentanglement into product states correspond to sufficient conditions for cloning of one of the subsystems [10], and the condition for disentanglement into separable states corresponds to a sufficient condition for broadcasting [11] of one of the subsystems.

Proposition 1. (a) Any set of perfectly distinguishable states can be disentangled. (b) Any set of states with identical reduced density matrices can be disentangled.

Proof. In the former case the disentanglement procedure consists of an identification of the state, followed by the bilocal preparation of the corresponding reduced density matrices. In the latter case, the same bilocal preparation is performed for any of the possible inputs. QED

A nice corollary of this result is that *any set of maximally entangled states can be disentangled*. It is so because a reduced density matrix of any maximally entangled state is a maximally mixed state, $\mathbf{1}/n$, where $\mathbf{1}$ is a unit matrix and n is a dimensionality of the subsystem.

It is interesting to note that the first part of the Proposition corresponds to the ''if'' part of the no-cloning theorems [10], namely, that perfectly distinguishable states can be cloned, and both parts correspond to the ''if'' part of the no-cloning theorem for mixed states [11], namely, that identical or orthogonal mixed states can be cloned.

To derive a sufficient condition for the disentanglement into separable states, we need to present here some properties of the broadcasting [11,12] of a quantum state onto two separate quantum systems. After the broadcasting the reduced density matrices of each of the subsystems are identical with the broadcasted state, which is destroyed. This procedure has at the input an unknown state ρ from the known list and an ancilla in some standard state Υ. Its output is some state $\tilde{\rho}$, $\rho\otimes\Upsilon\rightarrow\tilde{\rho}$, which satisfies

$$\mathrm{tr}_A\,\tilde{\rho} = \mathrm{tr}_B\,\tilde{\rho} = \rho. \tag{6}$$

A necessary and sufficient condition for the broadcasting of a set is that the density matrices of its states commute.

For a perfect disentanglement we need only the ''if'' part of this result. If density matrices commute, they can be diagonalized simultaneously; therefore, we write them in their eigenbasis. The standard ancilla state is taken to be $\Upsilon = |0\rangle\langle 0|$. As a result, any unitary transformation which is consistent with

$$|0\rangle|0\rangle\rightarrow|0\rangle|0\rangle \quad \text{and} \quad |1\rangle|0\rangle\rightarrow|1\rangle|1\rangle \tag{7}$$

does the job. In particular,

$$\mathbf{U} = \mathbf{U}^{\dagger} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{8}$$

is such a transformation. Now we are able to prove the following

Proposition 2. Any set of entangled qubits can be disentangled into separable states if the reduced density matrices of one of the parties (Alice or Bob) commute.

Proof. Let us suppose that Bob's reduced density matrices commute and chose the local basis where they are diagonal. Then the most general form of a density matrix that can belong to this set is

$$\rho^{AB} = \begin{pmatrix} a & b & c & d \\ b^* & e & f & g \\ c^* & f^* & h & -b \\ d^* & g^* & -b^* & s \end{pmatrix}, \tag{9}$$

where $s = 1 - a - e - h$ and the matrix is a subject to the positivity constraints. Bob's reduced density matrix is

$$\rho^{B} = \begin{pmatrix} a+h & 0 \\ 0 & e+s \end{pmatrix}. \tag{10}$$

We append the ancilla C in the state $|0\rangle\langle 0|$ at Bob's side and perform a local broadcasting [with the operator $\mathbf{U}$ of Eq. (8) with Bob's bit first (control bit) and the ancilla bit second (target bit)],

$$\rho^{AB}\otimes|0\rangle\langle 0|\rightarrow\tilde{\rho}^{ABC} = (\mathbf{1}^{A}\otimes\mathbf{U}^{BC})(\rho^{AB}\otimes|0\rangle\langle 0|)(\mathbf{1}^{A}\otimes\mathbf{U}^{BC}). \tag{11}$$

Now we can trace out either Bob's particle or the ancilla. The result is the same and the output is

$$\rho_{out} = \begin{pmatrix} a & 0 & c & 0 \\ 0 & e & 0 & g \\ c^* & 0 & h & 0 \\ 0 & g^* & 0 & s \end{pmatrix}, \tag{12}$$

which obviously has the same reduced density matrices as its original from the Eq. (9).

It remains to show that ρ_{out} is a separable state. For 2×2 density matrices a positive partial transposition is a necessary and sufficient condition of a separability [13]. Namely, a density matrix ρ is separable if and only if the matrix which results from its partial transposition is positive, i.e., represents a valid physical state.

We perform a partial transposition on the second system and get

$$\sigma=\rho_{out}^{T_2}=\rho_{out}, \tag{13}$$

which is identical with the original density matrix. Thus ρ_{out} is always separable. QED

It is easy to see why neither of the above sufficient conditions is not a necessary one. For example, reduced density matrices of the states that are already separable need not to commute. However, the procedure of their ''disentanglement'' consists in doing nothing—$\mathbf{U}=\mathbf{1}$.

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