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Multi-particle entanglement via two-party entanglement*

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Abstract

Entanglement between n particles is a generalization of the entanglement between two particles, and a state is considered entangled if it cannot be written as a mixture of tensor products of the n particles' states. We present the key notion of *semi-separability*, used to investigate n-particle entanglement by looking at two-party entanglement between its various subsystems. We provide *necessary conditions* for n-particle separability (that is, *sufficient conditions* for n-particle entanglement). We also provide necessary and sufficient conditions in the case of pure states. By surprising examples, we show that such conditions are not sufficient for separability in the case of mixed states, suggesting entanglement of a strange type.

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1. Introduction

Entanglement between two particles A and B provides correlations that have no classical counterpart [7, 2, 11]. A two-particle pure state can either be a tensor product of the one-particle states, or else it contains entanglement between them. In general, the state of a composite system is a mixed state: a mixture of pure states. For mixed states, in addition to being entangled or a tensor product, there is a third possibility, which is a mixture of tensor-product states. Such a state is not entangled, and the name *separable state* is used to consider both tensor products and mixtures of tensor products. Fulfilling Bell's inequalities provides necessary conditions for separability. (Equivalently, breaking Bell's inequalities provides sufficient conditions for entanglement.) There are many other necessary conditions for separability, such as the inability for the state to be used to teleport a qubit [3]. The other direction is much more subtle; necessary conditions for two-particle entanglement (that

* A preliminary version of this work was presented at the First NASA QCQC'98 conference [1].

is, sufficient conditions for separability) were given in some special cases [12], but are still missing in the general two-particle case [9].

For three and more particles, the situation is more complex as we demonstrate in this paper. Entanglement between n particles is a generalization of the entanglement between two particles, and a state is considered entangled if it cannot be written as a mixture of tensor products of the n particles' states. We discuss properties which, by definition, are non-trivial only for systems composed of more than two particles, such as two-party entanglement or separability, which is entanglement or separability between the various possible pairs of subsystems. For instance, there are six possibilities of forming such pairs in the three-particle (A, B, C) case: $(A; B)$, $(B; C)$, and $(C; A)$ when one particle is traced out and the other two are given to the two parties, and $(AB; C)$, $(BC; A)$, and $(CA; B)$ when two particles are given to one party and the third particle to the other party.

2. Semi-separability

We define *k-party semi-separability* of a system as follows. Let a system contain n particles such that $n > k$. Divide all particles into k or $k+1$ non-empty groups of particles (k or $k+1$ subsystems). If we divided into $k+1$ subsystems, trace out one of them to be left with k subsystems in any case. The original system is semi-separable if those k subsystems are separable.

We shall distinguish the two cases, depending on whether we trace out one subsystem or not.

***k*-party total semi-separability.** Divide all particles into k non-empty groups (k subsystems) and check separability of the k subsystems.

***k*-party partial semi-separability.** Divide all particles into $k + 1$ non-empty groups ($k + 1$ subsystems), trace out one subsystem, and check separability of the remaining k subsystems.

We shall refer to these as *k-TSS* and *k-PSS*, respectively. The case $k = 2$ is of special interest since the separability of two parties (two subsystems in our case) has been extensively studied. We concentrate on $k = 2$ in the following and drop the index k unless there is a danger of confusion.

We use these properties to provide a partial classification of n-particle entanglement and separability in terms of the much simpler (albeit still only partially solved) problem of separability of two subsystems, each possibly composed of several of the original particles. We provide *necessary conditions* for n-particle separability in the general case. We also provide necessary and sufficient conditions in the case of pure states, conditions which do not hold for mixed states. These properties provide a new insight into many-particle entanglement and allow us to find a surprising type of entanglement, which shows a new fundamental difference between the properties of pure states and of mixed states.

These properties may be useful in the future in providing a complete classification of separability versus entanglement of many particles. Different and independent approaches to many-particle entanglement have been discussed in other works [1, 10, 6].

3. Conditions for multi-particle separability

Most of the discussion in this section is restricted (for simplicity) to two-subsystem entanglements in a system composed of three particles, but is true for two subsystems in a system composed of n particles unless explicitly stated otherwise. Furthermore, we restrict our examples to qubits, to make it simple and clear, but *all* the *Facts* we provide below apply to higher-dimensional Hilbert spaces.

Let Alice, Bob and Carol (who are spatially separated) have three qubits (denoted by A, B and C respectively), prepared in some joint pure state $|\Psi^{(r)}\rangle$, where (r) is used to index one of possibly many such states:

$$|\Psi^{(r)}\rangle = \sum_{i=0}^{7} \alpha_i^{(r)} |i\rangle = \alpha_{000}^{(r)} |0_A 0_B 0_C\rangle + \alpha_{001}^{(r)} |0_A 0_B 1_C\rangle + \cdots + \alpha_{111}^{(r)} |1_A 1_B 1_C\rangle \tag{1}$$

with $\sum_i |\alpha_i|^2 = 1$. The states $|0\rangle$ and $|1\rangle$ are basis vectors of each qubit, and $|00\rangle \equiv |0\rangle \otimes |0\rangle$. We usually avoid using the tensor product notation $\otimes$ unless we want to emphasize it. The most general three-qubit state is a mixture of states of this type

$$\rho = \sum_r p_r |\Psi^{(r)}\rangle\langle\Psi^{(r)}| \tag{2}$$

with $\sum_r p_r = 1$. Any such state can be written in various forms. A three-particle state is considered separable, or non-entangled (NE), if and only if it can be written as a mixture of tensor products

$$\rho_{\mathrm{NE}} = \sum_s p_s \left[\rho_A^{(s)} \otimes \rho_B^{(s)} \otimes \rho_C^{(s)}\right]. \tag{3}$$

A *pure state* can be presented using its Schmidt decomposition [11], provided we separate one subsystem at a time (and do it recursively if there are more particles). A separable pure state is necessarily in a tensor product of one-particle pure states $|\Psi_{\mathrm{NE}}\rangle = |\Psi_{AB}\rangle \otimes |\phi_C\rangle = |\phi_A\rangle \otimes |\phi_B\rangle \otimes |\phi_C\rangle$ where the $|\phi\rangle$'s are one-qubit states.

Let us study a three-particle entangled state by looking at the entanglement between its various two subsystems. There are three options for partial semi-separability: tracing out (ignoring) one particle to be left with two particles. Similarly, there are three options for total semi-separability: considering two particles as one subsystem, to be left with this combined subsystem and the remaining particle. If we trace out particle C and the remaining state of systems A and B is separable, we denote this two-subsystem partial semi-separability by $PSS_C(A; B)$. If the subsystem composed of AB is separable from C, we denote this two-subsystem total semi-separability[1] by $TSS(AB; C)$.

Similar notation can be written for the other four options obtained by cyclic permutations of the three particles. A negation is denoted by $\overline{TSS}\,(AB; C)$ saying that AB is entangled with C. In the general case of n particles and k subsystems, more terms could appear; for instance, 4-$TSS(A; BC; D; E)$ and 3-$PSS_{CD}(A; B; E)$. Once these new notions are established, we can use them to prove some simple facts, relating n-particle entanglement to the better understood entanglement between two systems.

A separable state (such as in equation (3)) presents all three possible *TSS* properties. This is immediately obtained from the fact that the state is still separable when collecting two particles together: for instance, collecting A and B together yields $\rho_{\mathrm{NE}} = \sum_s p_s[\rho_{AB}^{(s)} \otimes \rho_C^{(s)}]$. Thus:

Fact 1. If a state does not present *all* cases of *TSS*, then it is entangled.

In other words, the existence of all possible *TSS* is a necessary condition for separability (and the existence of one $\overline{TSS}$ is a sufficient condition for entanglement). In the original conference

[1] Later works, which appeared after the 1998 preliminary version of ours [5], refer to this as *bi-separability along the* $(AB; C)$ *cut*.

version of this paper [5], we had conjectured that the converse should hold because it seemed reasonable to expect that a system that shows all cases of *TSS* should in fact be separable. This conjecture was subsequently proven wrong via an explicit counter example [4], which is provided here in section 5 for the sake of completeness. The conjecture proved to be very fruitful because its refutation was followed by an extensive research on the bi-separability of multi-particle systems.

A separable state (such as in equation (3)) satisfies

$$\mathrm{Tr}_C\,\rho_{\mathrm{NE}} = \sum_s p_s \mathrm{Tr}_C\big[\rho_A^{(s)} \otimes \rho_B^{(s)} \otimes \rho_C^{(s)}\big] = \sum_s p_s\big[\rho_A^{(s)} \otimes \rho_B^{(s)}\big] \qquad (4)$$

due to the linearity of the trace-out operation. Similar equations can be written if A or B are traced out. Therefore it presents all *PSS* properties. Thus:

Fact 2. If a state does not present *all* cases of *PSS*, then it is entangled.

In other words, the existence of all possible *PSS* is a necessary condition for separability (and the existence of one $\overline{PSS}$ is a sufficient condition for entanglement). The converse is not true and a three-particle *entangled state* might present all *PSS* properties: the GHZ–Mermin state [7] is a good example—see section 4. Therefore, the existence of *all PSS* is a necessary condition for separability, but not a sufficient one.

Note that equation (4) also implies that if a state fulfils $TSS(A; BC)$ then automatically it also fulfills $PSS_C(A; B)$ and $PSS_B(A; C)$, and similar conclusions can be obtained by cyclic permutations. Thus, if a state fulfils two total-semi-separability conditions, say $TSS(A; BC)$ and $TSS(B; CA)$, then it fulfils all three *PSS* conditions.

For pure states, we now obtain conditions that are both necessary and sufficient for separability. (We need only prove that they are sufficient, since the fact that they are necessary is already shown using the previous facts.) We omit Dirac's braket notation unless confusion could arise.

Fact 3. A pure state that presents all *TSS* properties (for all possible decompositions of two subsystems) is separable.

Proof. We prove Fact 3 in the case of three particles. If the three-particle pure state is totally semi-separable for all possible ways of decomposing the subsystems then it can be written as $\Psi_{ABC} = \Psi_{AB} \otimes \Psi_C$. If Ψ_{AB} can be written as $\Psi_A \otimes \Psi_B$ then $\Psi_{ABC} = \Psi_A \otimes \Psi_B \otimes \Psi_C$ and the state is separable.

Since Ψ_{ABC} presents all *TSS* properties it can also be written as $\Psi_{ABC} = \Phi_A \otimes \Phi_{BC}$. Assuming for a contradiction that Ψ_{AB} cannot be written as $\Psi_A \otimes \Psi_B$, it can be decomposed (due to Schmidt decomposition [11]) as $\alpha\Psi_A \otimes \Psi_B + \beta\Psi'_A \otimes \Psi'_B$ with $\alpha \neq 0$ and $\beta \neq 0$, where χ' denotes an orthogonal state to χ for any state χ. Now $\Psi_{ABC} = \alpha\Psi_A \otimes (\Psi_B \otimes \Psi_C) + \beta\Psi'_A \otimes (\Psi'_B \otimes \Psi_C)$ is (by treating B and C together) the Schmidt decomposition of A and BC, showing entanglement between A and BC, in contradiction to $\Psi_{ABC} = \Phi_A \otimes \Phi_{BC}$. **QED**

Fact 3 can be strengthened. In the case of three particles, we have:

Fact 3′. A three-particle pure state that is *TSS twice* is separable.

Proof. Without loss of generality, suppose that the two *TSS* properties are $TSS(AB; C)$ and $TSS(A; BC)$. We can make use of the proof of Fact 3 since it used only these two semi-separabilities anyhow. **QED**

Fact 4. A pure state that presents a particular *TSS* in which one particle is separable from all the others, and also presents all *PSS*, is separable.

Proof. Again, we prove it in the case of three particles. Since the three-particle pure state is totally semi-separable once (say, *TSS*(*AB*; *C*)) it can be written as $\Psi_{ABC} = \Psi_{AB} \otimes \Psi_C$. We now trace out particle *C* to get $\Psi_{AB} = \Psi_A \otimes \Psi_B$ since the state presents all *PSS*. Thus, $\Psi_{ABC} = \Psi_A \otimes \Psi_B \otimes \Psi_C$. **QED**

We conclude that Facts 3, $3'$ and 4 provide necessary and sufficient conditions for the separability of pure states.

Surprisingly, Fact 4 is not true for mixed states. We present in section 5 a three-particle *mixed state* that presents one case of *TSS* (say, *TSS*(*A*; *BC*)) and all *PSS*, yet it is entangled. More surprisingly, Fact 3 is also not true for mixed states. We present in section 5 a three-particle *mixed state* that presents all cases of *TSS*, yet it is entangled [4]; such a state has a special property—it is undistillable (meaning that no arbitrarily pure singlet state can be obtained from it, even if an unbounded number of copies is available).

4. A few simple examples

Let us first provide some examples of pure states in order to obtain a better intuition about the meaning of the different semi-separability conditions. We also describe the techniques required for verifying the existence of semi-separabilities.

Our first example is

$$|\Psi_{En}\rangle = |\Psi_A\rangle \otimes |\Psi^-_{BC}\rangle \tag{5}$$

with $|\Psi^-\rangle = (1/\sqrt{2})[|01\rangle - |10\rangle]$ is the singlet state of two qubits. When the non-entangled particle is traced out, the other two are still entangled so we have $\overline{PSS}_A(B; C)$, but when *B* or *C* are traced out the remaining particles are not entangled, so we have $PSS_B(A; C)$ and PSS_C $(A; B)$. Clearly it is also *TSS(A; BC)*, $\overline{TSS}(AB; C)$ and $\overline{TSS}(AC; B)$: When the two particles *A* and *B* are considered as one subsystem *AB*, it is fully entangled with subsystem *C* since the state can be written as $(1/\sqrt{2})[|\Psi_A 0\rangle \otimes |1\rangle - |\Psi_A 1\rangle \otimes |0\rangle]$, and the states $|\Psi_A 0\rangle$ and $|\Psi_A 1\rangle$ play the role of the relevant two basis vectors $|0\rangle$ and $|1\rangle$ of the four-dimensional subsystem.

Consider a three-qubit case. If we write the three-particle general pure state as

$$|\Psi\rangle = |0_A\rangle \left(\sum_{k=0}^{3} \alpha_{0k}|k_{BC}\rangle\right) + |1_A\rangle \left(\sum_{k=0}^{3} \alpha_{1k}|k_{BC}\rangle\right) \tag{6}$$

then tracing out subsystem *A* will leave the other two subsystems in a (possibly mixed) state

$$\begin{aligned}\rho &= \left(\sum_{k=0}^{3} \alpha_{0k}|k_{BC}\rangle\right)\left(\sum_{l=0}^{3} \alpha^*_{0l}\langle l_{BC}|\right) + \left(\sum_{k=0}^{3} \alpha_{1k}|k_{BC}\rangle\right)\left(\sum_{l=0}^{3} \alpha^*_{1l}\langle l_{BC}|\right) \\ &= \sum_{k=0}^{3}\sum_{l=0}^{3}(\alpha_{0k}\alpha^*_{0l} + \alpha_{1k}\alpha^*_{1l})|k\rangle\langle l|. \end{aligned} \tag{7}$$

Using the partial transposition technique of Peres [12], it is possible to check whether or not this reduced density matrix is separable for any particular case. This result (and similar results for tracing out the other particles) can be used to verify if entanglement between two particles exists. For larger systems (such as qutrits, which are three-dimensional Hilbert spaces), no perfect way to tell if the two remaining qutrits are separable has been found so far, and it is known that Peres's criterion does not apply [9].

To verify that all cases of *PSS* are present, the three possibilities must be checked. For instance, the state

$$|\Psi_{ABC}\rangle = \frac{\cos\theta}{\sqrt{2}}|000\rangle + \frac{\cos\theta}{\sqrt{2}}|011\rangle + \frac{\sin\theta}{\sqrt{2}}|100\rangle - \frac{\sin\theta}{\sqrt{2}}|111\rangle \tag{8}$$

(with $|000\rangle \equiv |0_A 0_B 0_C\rangle$) leads to the reduced density matrix

$$\rho_{BC} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & \cos 2\theta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \cos 2\theta & 0 & 0 & 1 \end{pmatrix} \tag{9}$$

which is entangled for $\theta \neq \pi/4$.

Checking the *TSS* property is sometimes impossible even for three qubits, using the best techniques currently available. If we write the three-particle general pure state of equation (6) using a Schmidt decomposition, it can be written as

$$|\Psi\rangle = \cos\theta|\phi_A\rangle|\chi_{BC}\rangle + \sin\theta \mathrm{e}^{\mathrm{i}\phi}|\phi'_A\rangle|\chi'_{BC}\rangle \tag{10}$$

(where $|\psi'\rangle$ stands for a state orthogonal to $|\psi\rangle$). Although it is not obvious to find this decomposition, this can always be done [11]. When both terms appear in the Schmidt decomposition, the state is $\overline{TSS}(A;\ BC)$. Generalizing this argument to mixed states is usually impossible with current techniques. Peres's partial transposition [12] works unconditionally [9] only for two qubits (2×2) and for a qubit and a qutrit (2×3), and it can only be used to check entanglement of a four-dimensional system with a two-dimensional system, but not to check separability.

In the following examples of pure states, the state is already given in such a Schmidt decomposition (for all possible combinations of pairs), so they clearly present no *TSS*. For a GHZ–Mermin state [8]

$$\Psi_{GHZM} = (1/\sqrt{2})[|0_A 0_B 0_C\rangle + |1_A 1_B 1_C\rangle] \tag{11}$$

let us consider the particles of Bob and Carol as one four-dimensional subsystem. Then this state becomes a Bell state

$$(1/\sqrt{2})[|0_A(00)_{BC}\rangle + 1_A(11)_{BC}\rangle] \tag{12}$$

so that $|00\rangle$ and $|11\rangle$ play the role of $|0\rangle$ and $|1\rangle$, the relevant two basis vectors of the four-dimensional subsystem. Clearly, the same is true for the other two cases. This state presents all *PSS*: when one particle is traced out, the other two are in a mixture-of-product state. Similar arguments apply to the following state:

$$\Psi_{Zei} = (1/\sqrt{2})[|0_A \Psi^+_{BC}\rangle + |1_A \Psi^-_{BC}\rangle] \tag{13}$$

with $|\Psi^+\rangle = (1/\sqrt{2})[|01\rangle + |10\rangle]$. Particle A is entangled to the subsystem composed of BC together. By transforming to the $|0 \pm 1\rangle$ basis of particle A, this state becomes a GHZ–Mermin state, so it presents all the same properties as above.

5. Surprising examples

We now present our first surprising example: a three-particle mixed state that presents one *TSS* and all *PSS*, but yet is entangled! (This should be contrasted with Fact 4.) Consider the state that is composed of an equal mixture of the two states $|\Psi_1\rangle = |0_A\rangle \otimes |\Psi^+_{BC}\rangle$ and $|\Psi_2\rangle = |1_A\rangle \otimes |\Psi^-_{BC}\rangle$, and thus

$$\rho = \tfrac{1}{2}[(|0_A\rangle\langle 0_A|) \otimes (|\Psi^+_{BC}\rangle\langle\Psi^+_{BC}|) + (|1_A\rangle\langle 1_A|) \otimes (|\Psi^-_{BC}\rangle\langle\Psi^-_{BC}|)]. \tag{14}$$

This state presents all *PSS*: when particle A is traced out, the other particles are left in an equal mixture of the two Bell states $|\Psi^+\rangle$ and $|\Psi^-\rangle$, which is separable since this is the same as an equal mixture of $|01\rangle$ and $|10\rangle$; when particle B (resp. C) is traced out, it is clear that the particle entangled to it, C (resp. B), does not become entangled with A. This state also presents $TSS(A; BC)$: when subsystems B and C are considered as one subsystem, A is clearly separable from it since they are written as a mixture of products.

To prove that it is an entangled state, let us show that it is not semi-separable for $TSS(AB; C)$. Then, Fact 1 implies that the three-particle state is entangled. The two pure states that are mixed to give ρ can be written as

$$|\Psi_1\rangle = |0_A 0_B\rangle \otimes |1_C\rangle + |0_A 1_B\rangle \otimes |0_C\rangle$$

and

$$|\Psi_2\rangle = |1_A 0_B\rangle \otimes |1_C\rangle - |1_A 1_B\rangle \otimes |0_C\rangle$$

up to normalization. Now we can see that C is entangled to different two-dimensional subspaces of AB in each of these pure states, thus mixing cannot reduce or cancel this entanglement. Such a state is surprising: although Alice is not entangled with the subsystem of Bob and Carol (together) she can *control* their entanglement. If she measures in the $|0 \pm 1\rangle$ basis, Bob and Carol will not be entangled whatsoever, but if she measures in the computational basis, Bob and Carol are entangled without knowing it, and their state depends on Alice's measurement result. Thus, Bob and Carol are in a separable state if they ignore (trace out) Alice's knowledge, but they become entangled once they receive Alice's result using classical communication.

We now present a second surprising example (originated in [4]): a three-particle mixed state that presents all *TSS* (and obviously, all *PSS* as well), but yet is entangled! (This should be contrasted with Fact 3.) Consider the state

$$\bar{\rho} = \frac{1}{4}\left(\mathbf{1} - \sum_{j=1}^{4} |\psi_j\rangle\langle\psi_j|\right) \tag{15}$$

with $\psi_1 = |0, 1, +\rangle$, $\psi_2 = |1, +, 0\rangle$, $\psi_3 = |+, 0, 1\rangle$ and $\psi_4 = |-, -, -\rangle$ where $\pm = (|0\rangle \pm |1\rangle)/\sqrt{2}$. The space complementary to

$$\{|0, 1, +\rangle, |1, +, 0\rangle, |+, 0, 1\rangle, |-, -, -\rangle\} \tag{16}$$

contains no product state orthogonal to these four states (see [4]), hence the state (15) is entangled.

The state (15) has the curious property that, even though it is entangled, it presents all cases of *TSS*: the entanglement across any split into two parties is zero. For example, to show $TSS(A; BC)$ (so that the entanglement between A and BC is zero), we write $a = |1, +\rangle$, $b = |+, 0\rangle$, $c = |0, 1\rangle$ and $d = |-, -\rangle$. Note that these are just the B and C parts of the four states in equation (16), and that $\{a, b\}$ are orthogonal to $\{c, d\}$. The vectors $a^\perp$ and $b^\perp$ in the span(a, b) and the vectors $c^\perp$ and $d^\perp$ in the span(c, d) can be used to complete the original set of vectors to a full product basis between A and BC with the states $\{|0, a^\perp\rangle, |1, b^\perp\rangle, |+, c^\perp\rangle, |-, d^\perp\rangle\}$. The state (15) is composed of these states, hence satisfies the desired *TSS* property. By the symmetry of the states, this is also true for the other splits.

6. Generalizations and conclusions

To summarize, we analysed three-particle entanglement/separability (and beyond) in terms of its possible two-subsystems entanglements/separability, which we call total and partial semi-separability. We presented necessary conditions for separability and also sufficient conditions in the case of pure states. We also discussed possible generalizations and presented surprising mixed entangled states.

Generalizations to larger systems are more complicated. A four-qubit system A, B, C and D, can be discussed in terms of two-particle/three-particle entanglement, and semi-separability to 2-$TSS(A; BCD)$, etc, and 2-$TSS(AB; CD)$, etc, and 3-$TSS(A; B; CD)$, etc, and also 3-$PSS_A(B; C; D)$, etc, and 2-$PSS_{AB}(C; D)$, etc, where all the 'etc' refer to permutations of the particles. A relatively simple (and yet interesting) example can be built by replacing $|0_A\rangle$ and $|1_A\rangle$ in the previous example of equation (14) by two Bell states of two particles (A and D):

$$\rho = \tfrac{1}{2}[(|\Psi^+_{AD}\rangle\langle\Psi^+_{AD}|) \otimes (|\Psi^+_{BC}\rangle\langle\Psi^+_{BC}|) + (|\Psi^-_{AD}\rangle\langle\Psi^-_{AD}|) \otimes (|\Psi^-_{BC}\rangle\langle\Psi^-_{BC}|)] \qquad (17)$$

so that Bob and Carol are entangled if Alice and David (D) measure their entanglement, but if Alice and David measure in the product basis $|00\rangle$, $|01\rangle$, etc, then Bob and Carol become disentangled.

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