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Automatic Reasoning for Pointer Programs Using Decidable Logics

Thesis submitted for the degree of Doctor of Philosophy

by

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Abstract

This thesis proposes a novel method addressing the verification problem for programs manipulating linked-list data structures and list-like variants such as reversed trees. It makes use of a restricted subset of first-order logic that is decidable, yet effective to support reasoning about paths of pointer-links in the program's dynamic heap. Such properties are essential for proving program termination, correctness of data structure invariants, and other safety properties. The core of the thesis is a complete axiomatization of transitive closure over uninterpreted functions embedded in effectively-propositional (EPR) first-order logic, such that existing SAT solvers can be harnessed to prove validity, or produce a concrete counterexample that falsifies a verification condition. Since this solution is logically complete and resides completely in a decidable set of first-order formulas, one of these two results is guaranteed — the procedure never diverges or returns an imprecise result.

We present techniques for modular reasoning. In a program with procedures, we address the problem of "global effect": a subroutine that changes a small area of the heap may affect reachability properties anywhere in the heap, because its operations remove existing paths and create new paths. This leads to some restrictions over what the callee may or may not do. Then, an adaptation rule is applied to incorporate mutations made by the callee into the caller's heap space.

Both analyses require the user to provide an appropriate inductive invariant for loops and for recursive procedures. The invariants may be nontrivial, and in particular, more complex than the specification of the program as a whole. To alleviate this problem, we employed an iterative refinement algorithm for automatic inference of inductive invariants over a set of abstraction predicates; The algorithm gradually constructs an over-approximation of the reachable states until it finds an inductive invariant that is sufficient to prove a desired safety property. This approach is known as "property-directed reachability". We showed that our implementation is capable of producing correct invariants for a set of benchmark programs and correctness properties.

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Chapter 1

Introduction

This thesis develops means for automated reasoning for the purpose of proving the correctness of computer programs making excessive use of pointers. These include programs manipulating linked data structures, such as linked lists, doubly-linked lists, nested lists, and reverse trees. For automation we use industry-standard tools whose efficiency has been proven in practice. The proposed logical frameworks reduces verification problems into logical queries. We show that the set of queries thus generated is a decidable one, so a definite answer ("yes" or "no") is guaranteed.

The problem of software verification is about as old as software, perhaps even older. Floyd and Hoare [19, 29] proposed logical frameworks to construct correctness proofs for programs with respect to formal specifications—*full correctness proofs*, which contain a proof for the program's termination on its designated set of input, as well as the adherence of its behavior to the one registered in the specification; and the more popular *partial correctness proofs*, which relax the termination requirement. A continued effort to mechanize the construction of such proofs existed ever since. Cousot and Cousot [10] brought the advancement of *abstract interpretation*, providing a multitude of techniques used to analyze software in various domains. Simple abstract interpretation techniques are employed day-to-day by compilers due to their elegance, ease of implementation, and good performance. Naturally, there is a trade-off between resource usage and accuracy of the analysis; as a consequence such analyses, which are based on simple abstractions, mostly provide approximate results.

Within the problem space of software verification, a particularly interesting subset of programs is those making heavy use of pointers. In C and the C-style programming languages that emerged as a result of its success, pointers are basic working tools

just like arithmetic operators and control structures, combining expressivity and low-level efficiency. *Separation logic*, mostly due to Reynolds [55] and O'Hearn [35, 50], has evolved to address this challenging aspect of programming. Challenges arise, by-and-large, by the occurrence of *aliasing* in pointer programs: the situation where two pointers contain the same address, hence a change in data stores at that address is visible at once in two places in the program. Consider the simple number-incrementing procedure (written in C):

```
void go_up(int *x, int *y) {
  (*x)++; (*y)++;
}
```

The programmer's intention is to increase both counters. However, in the corner case where the parameters `x` and `y` store the same address (are aliased), the result would be increasing one counter by two. If the programmer did not plan this scenario, it could lead to intricate hard-to-find bugs at runtime. An attempt to formally prove the correctness of this procedure would immediately give rise to a candidate specification, such as the one expressed by the equations:

$$[x] = \underline{[x]} + 1$$
$$[y] = \underline{[y]} + 1$$

Here we employ a somewhat standard notation where $\underline{x}$ denotes the *input* value of the program variable `x`, and plain x denotes its *output* value. The square brackets indicate that the equation holds on the values stored in the address given by the variable, not on the addresses themselves.

From here it is plain to see the defect: in the case where $x = y$ (aliasing) the specification contradicts the program. This is because the specification is *declarative*, so the order of equations is insignificant and in fact the two equalities state the same property: that the memory location referenced by both x and y has its value increased by one. In the program's operational semantics, however, the fact that the assignment repeats itself is, of course, significant—the value will be increased by two. Using a formal semantics of the language would systematically reveal this discrepancy.

Complex composition of pointers leads to more involved reasoning. The program shown in Fig. 1.1 is an early example used by Reynolds. The program reverses the

$$j := \text{null} \ ; \ \textsf{while } i \neq \text{null } \textsf{do}$$
$$(k := i.next \ ; \ i.next := j \ ; \ j := i \ ; \ i := k)$$

Figure 1.1: A classical program that performs in-place reversal of a list, adapted from [55]

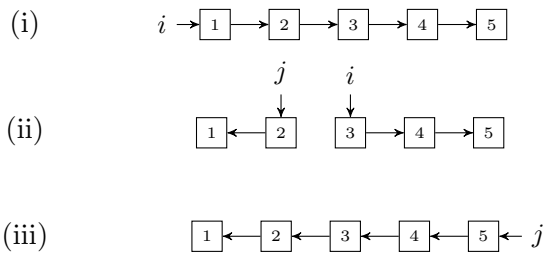

Figure 1.2: The state of the memory during the execution of the list reversal program: (i) initial state; (ii) an intermediate state; (iii) final state.

order of elements in a linked list: its input is a linked list whose head is pointed to by i and each node contains a field *next* holding a pointer to the next node (or null to signify the last node). The program outputs a list of the same structure, only that the elements occur in reversed order. The reversal is done in-place, so that the original input list is overwritten by the output.

Reynolds identified an acute problem when reasoning with programs that traverse such recursive data structures: the pointer i serving as the iterator is advanced at each step, and the number of steps is not bounded. Therefore there is always a risk that a value written on one iteration will be overwritten in subsequent iterations. In particular, to make sure that the list remains acyclic in this example, one must obtain that there is no *next*-path from j to i, otherwise the addition of the edge $\langle i, j \rangle$ introduces a cycle.

We approach this problem by a careful construction of appropriate loop invariants for iterative programs, and comprehensive summaries for recursive programs. Observing a typical run of *reverse* (Fig. 1.2), an important property of it can be noticed: the pointer variables i and j always point into the beginning of two **disjoint** list segments. Either segment may be empty (as in (i) and (iii)), but the segments never share elements. It turns out that this property is crucial to prove the correctness of the list reversal program. Formulating this property in logic is more involved than the previous, simpler aliasing conditions. To address this issue, we define *reachability logics* and

support reasoning to check the validity of implications. In this approach, we would write an invariant such as

$$\forall \alpha : i\langle next^* \rangle \alpha \wedge j\langle next^* \rangle \alpha \rightarrow \alpha = \text{null} \tag{1.1}$$

The concern raised by Reynolds was that such an approach would never scale. To alleviate this, we suggest breaking the program down into small pieces where the properties are simple enough, then combining the sub-proofs to verify the whole program. We show that for many naturally occurring instances the invariants are quite managable and automatic reasoning is tractable. To scale up we continue to develop logical tools for modular reasoning. We return to discuss the *reverse* example in much detail in Section 3.

In this thesis, we draw primarily on the development of Hoare logic and its extensions, generally referred to as "Hoare-style verification". Hoare logic is a proof system for reasoning about programs with respect to their specifications, given as assertions—generally pre-condition and post-condition—written in a logical language of choice. While complete systems exist for Turing-complete imperative programming languages (one example is presented in the preliminaries of this thesis), the problem of proof search is a primary obstacle to implementing automated verification and program reasoning systems. This is true even for very small programs, since a proof is required to use formulas (assertions) that do not occur in the specification or in the program itself (such as Eq (1.1) above). Even when these are given, verifying that they construct a valid proof is an undecided problem, in general, since it requires proving the validity of formulas in the assertion language. The problem is especially difficult when programs include loops. A loop in the program can cause a code block to be executed arbitrarily many times, so that the number of states a program visits during its execution is disproportionate to the size of the program. Reasoning about sets of states is, in general, a higher-order problem.

This thesis attempts to greatly simplify reasoning by reducing the proof obligations that need to be checked to proving validity of sentences in propositional calculus. Thus we show that the program's assertions are correct if and only if some propositional formula is valid—or equivalently, its negation is unsatisfiable. Boolean satisfiability is a decidable problem, which is usually solved by dedicated software known as SAT

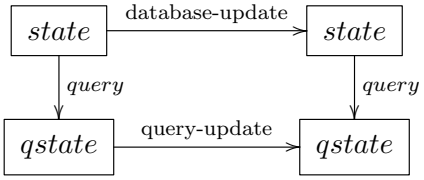

Figure 1.3: The view update problem, naturally occurring in databases but also applies to heap reachability.

solvers. A lot of engineering effort went into this kind of software over the years, and in practice, such instances are solved very efficiently and effectively.

Two central observations underpin our method. (i) In programs that manipulate singly- and doubly-linked lists it is possible to express the 'next' pointer in terms of the reachability relation between list elements. This permits direct use of recent results in descriptive complexity [28]: we can maintain reachability with respect to heap mutation in a precise manner. Moreover, we can axiomatize reachability using quantifier-free formulas. (ii) In order to handle statements that traverse the heap, we allow verification conditions (VCs) with $\forall^*\exists^*$ formulas so that they can be discharged by SAT solvers (as we explain shortly). However, we allow the programmer to only write assertions in a restricted fragment of FOL that disallows formulas with quantifier alternations but allows reflexive transitive closure. The main reason is that invariants occur both in the antecedent and in the consequent of the VC for loops; thus the assertion language has to be closed under negation, although the verification conditions are not required to have this property.

The appeal to descriptive complexity stems from the fact that previously it has been applied to the view-update problem in databases. This problem has a pleasant parallel to the heap reachability update problem we are considering. In the view-update problem, the logical complexity of updating a query w.r.t. database modifications is lower than computing the query for the updated database from scratch (depicted in Fig. 1.3). Indeed, the latter uses formulas with transitive closure, while the former uses quantifier-free formulas without transitive closure. In our setting, we compute reachability relations instead of queries. We exploit the fact that the logical complexity of adapting the (old) reachability relation to the updated heap is lower than computing the new reachability relation from scratch. The solution we employ is similar to the use of dynamic graph algorithms for solving the view-update problem, where directed paths between nodes are updated when edges are added/removed (e.g., see [13]), except that

our solution is geared towards verification of heap-manipulating programs with linked data structures.

Another aspect that complicates programmatic reasoning, especially with complex states as is the case when pointer-based data structures are present, is procedures. Programs are usually factored into several sub-programs in the interest of readability and code reuse. This common idiom causes one code block to be executed in different contexts, and it is highly desirable for reasons of scalability not to have to verify it for each context separately. The challenge is to be able to express the view-update that summarizes the effect of a procedure call in an arbitrary context, where some of the elements are not reachable by the procedure, and therefore essentially remain unmodified. This may be seen as an instance of the *frame problem*.

The rest of this thesis introduces *reachability logics*, a formal definition of logical fragments found useful for the systematic reasoning over programs containing pointer structures. As a primary technique, the semantics of such logics are embedded in first-order logic for the use of automated solvers. While a severe limitation on the expressivity of the defined logic, automated proof techniques prove to be so effective compared to manual proofs, even for small, seemingly-obvious examples, that there is much benefit to using them whenever possible.

The results in this thesis were published in [36], [38], and [37].

1.1 Main Results

1.1.1 Deterministic Transitive Closure

A key to the reasoning techniques presented in this thesis is the concept of *deterministic transitive closure* and its introduction into first-order logic. We begin by defining the notation $\langle next^* \rangle$, which denotes the reflexive transitive closure of a unary function symbol $next$. The semantics are that $x\langle next^* \rangle y$ is **true** iff there is a sequence of successive applications $next$ to x that results in y ($next(next(\cdots x \cdots)) = y$). The restriction that transitive closure can only be applied to functions is what makes it deterministic. As we will see, this form of transitive closure is much simpler to handle. This has been noticed before, in other contexts, e.g. in [34].

It is then shown that $\langle next^* \rangle$ can be axiomatized in pure first-order logic, in the same way that first-order logic with equality can be axiomatized in first-order logic

(without equality) by adding the equality axioms. One important restriction, however, is that once the transitive closure is introduced, the original function cannot be used in logical terms anymore— because the relationship between *next* and $next^*$ is not first-order-expressible. While this seems severe, it turns out that many useful properties of linked lists and some other linked data structures can be expressed this way. It is analogous to reasoning about natural numbers without using *succ* (the successor function) but with the relation $\leq$. In fact, it is easy to see that using quantification one can define *succ* in terms of $\leq$:

$$succ(x) = y \quad \Leftrightarrow \quad \forall \alpha : (x \leq \alpha \wedge \alpha \neq x) \leftrightarrow y \leq \alpha \tag{1.2}$$

This leads to a second restriction: the axiomatization of $\langle next^* \rangle$ that we construct in Section 3.2 is complete, but only for finite structures; hence, it is essential that the logic used for reasoning has the *finite model property*: if a formula in the logic has a model, then it also has a finite model. This is not true in general, of course, for first-order formulas. One fragment of first-order logic that does have this property will receive a lot of attention throughout this thesis is the Bernays-Schönfinkel-Ramsey class— also referred to as *effectively propositional* (EPR). It is characterized by a relational vocabulary, that is, only relation symbols and constants may occur in the signature and no non-nullary function symbols, and a quantifier prefix limited to $\exists^*\forall^*$ so that all existential quantifiers precede the universal ones. Again, we show a range of benchmarks demonstrating specifications that fall well within this restriction. In fact, most of the time, just universal formulas suffice to express desired properties.

In particular, it is very important to be able to express in a precise manner the effect of heap mutations performed by the program. A commonly used solution is to model the heap as a long array of pointers and to use McCarthy's axioms defining the update of an array a at index i to the value e as $a\{i \leftarrow e\}$. However, the use of transitive closure will lead us to expressions of the form $\langle (next\{i \leftarrow e\})^* \rangle$, which cannot be handled using the decidable logical fragment that we are interested in.

The proposed solution is to model the heap abstractly as a directed graph, and the transitive closure of the edges as a view of this graph. It is then our task to maintain the view across changes to the edge set of the graph. As a very basic example, Fig. 1.4(i) shows a snapshot of a heap containing two linked lists. The *next* edge set is $\{\langle 1, 2\rangle, \langle 3, 4\rangle, \langle 4, 5\rangle\}$ and the transitive closure $next^*$ is the binary relation

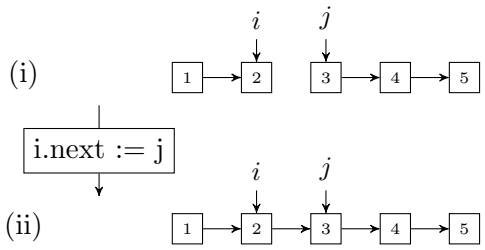

Figure 1.4: The effect of an assignment statement on the heap viewed as a directed graph.

$\{\langle 1,2\rangle, \langle 3,4\rangle, \langle 3,5\rangle, \langle 4,5\rangle\}$. The depicted assignment statement i.n := j causes the insertion of a new edge $\langle 2,3\rangle$, which materializes many new paths in the view of $next^*$, in particular: $\langle 1,3\rangle, \langle 1,4\rangle, \langle 1,5\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 2,5\rangle$ (as shown in Fig. 1.4(ii)); that is, all the pairs where the first element belongs to the first list $1 \to 2$ and the second element belongs to the second list $3 \to 4 \to 5$. This can be formulated in logic as a view update:

$$\alpha\langle next^*\rangle\beta \;:=\; \alpha\langle next^*\rangle\beta \vee \big(\alpha\langle next^*\rangle i \wedge j\langle next^*\rangle\beta\big) \tag{1.3}$$

This shows that $next^*$ can be updated using only previous values of $next^*$. Moreover, this particular update is quantifier-free, which plays well with the quantifier prefix limitations of EPR explained before. Chapter 3 and Chapter 5 extensively investigate the capability to express various kinds of view updates in a logic with restricted vocabulary and quantifiers.

1.1.2 Idempotent Functions in EPR

The use of EPR strictly rules out any function symbols other than constants. As a corollary, we point out an interesting case where the use of a function is benign—in the case where there is only one function symbol, and this function is *idempotent*, that is, it is deducible from the axioms that $\forall\alpha : f(f(\alpha)) = f(\alpha)$. In this case we show that there is a reduction from the satisfiability problem of formulas in the language that includes f to satisfiability of EPR formulas without functions, but with the addition of a finite number of constants and variables. The increase in formula size is linear in the number of symbols used. Therefore, adding such a function symbol f does not break the decidability property of the logical fragment. This extension can be used to write formulas in a more natural way.

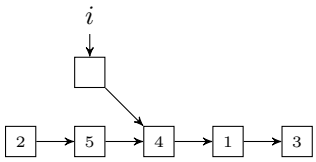

Figure 1.5: An example of a cutpoint into a linked list.

1.1.3 Procedural Reasoning with Adaptation

When dealing with a procedure that manipulates a linked-list segment or several linked-list segments, the most difficult issue is the existence of *cutpoints* [56], which are pointers from other areas of the heap that reach nodes of the list being manipulated. Such pointers are represented in our abstraction of the heap as edges, which the procedure does not modify and in fact may not even be able to observe, but may participate in paths that are being connected or disconnected by it. As an example, assume that the numbered list in Fig. 1.5 is being sorted by some sorting procedure. The node pointed to by i is connected to the node 4 by a pointer field. Before the sorting, the set of nodes reachable from i is, as seen in the figure, $\{4, 1, 3\}$. However, once the list is reordered according to the numbering, the reachable set would become $\{4, 5\}$. This is despite the fact that the sorting routine does not change i or the outgoing edge.

To support such reasoning, we define the notion of a *mod-set*, which is the area of the heap being directly mutated by the procedure, and an *entry function* used to describe the cutpoints by associating every "foreign" node from outside the mod-set with the first location in the mod-set that is reachable from that node. This allows us to formulate an *adaptation rule* that determines how the reachable set is modified for every node outside the mod-set, essentially providing a complete, precise characterization of the reachability relation $\langle next^* \rangle$ for the entire heap.

Having observed the behavior of several heap-manipulating procedures, we found that they share a common desirable property: the amount of shared location introduced by a single function call is bounded, and this bound can be known at compile time. We take advantage of this property in order to produce modular verification conditions that do not contain quantifier alternation, thus supporting our propositional reasoning based on the small-model property.

We use the fact that the entry function is idempotent to support decidable reasoning using the adaptation rule within the scope of EPR as explained in the previous

paragraph. Thanks to this property, verification rules generated for the modular case are still decidable for validity.

1.1.4 Template-based Verification with Induction

The downside of essentially every Floyd-Hoare verification technique is that it requires analyzed programs to be annotated with a *loop invariant* for every loop that occurs in the program. Furthermore, the supplied invariants should satisfy the restriction that they are *inductive*, that is, any single loop iteration should preserve their validity by induction: if invariant I holds at iteration k, we must be able to conclude that it holds at iteration $k+1$ (if such an iteration exists, of course) without knowing anything else about the state. This puts a heavy burden on a programmer who wishes to verify her program using such a method, since the discovery of an appropriate inductive invariant is usually not very intuitive. Furthermore, a small change in the program or in its specifications may require considerable change in the loop invariant.

We address the problem of *inductive invariant inference* by proposing a set of logical templates to serve as building blocks for a space of candidate invariants. For example, the following formulas are among the ones used as templates:

$$x = y \qquad x\langle next\rangle y \qquad x\langle next^*\rangle y \qquad \forall\alpha : x\langle next^*\rangle\alpha \wedge y\langle next^*\rangle\alpha \rightarrow \alpha = \text{null}$$

The symbols x and y serve as placeholders to be replaced by program variables. Based on our experiment, we constructed the search space as all possible Boolean combinations of template instances. In the example program *reverse*, there are 4 program variables; therefore 16 different assignments for x and y, giving a total of 64 instances. To take into account Boolean combinations of these, each of them may occur positive or negated ($\neg$), so the basic "literals" are 128, from which 2^{128} disjunctive *clauses* may be formed. The number of combinations is therefore the number of possible conjunctions, $2^{2^{128}}$.

Undoubtedly, this search space is much too vast for a naïve generate-and-test approach. Fortunately, there are search techniques far superior. Recent developments in model checking have given rise to the IC3 algorithm for inferring inductive invariants given safety properties, which are usually non-inductive; IC3 can be used to find a stronger property—that is, one that entails safety—that is inductive. It is much more

efficient than an explicit exploration of the state space needed in order to discover all the reachable states. Although originally developed to verify hardware systems, such as Boolean circuits, where the state is represented as a string of bits of a known length, we have found that it combines well with *predicate abstraction*, which allows to extend its use to software systems with unbounded state resulting in infinitely many states.

We have effectively applied IC3 with predicate abstraction for the domain of linked-list programs containing loops. The IC3 routine requires a theory solver, and the EPR-based encoding of deterministic transitive closure makes a perfect match. To use it, we made sure that all the abstraction predicates are expressible as universal formulas in the language containing $\langle next^* \rangle$, and by careful construction of the two-vocabulary formula $\rho(X, X')$, where X is a set of "state symbols", which are all the program variables + the special symbol $next^*$, and X' are primed versions of all these symbols, we obtained an instance of IC3 where all the satisfiability queries encountered during its execution are of EPR formulas. This ensures termination and relative completeness of the implementation. That is, if an inductive strengthening of the safety property exists and is expressible as a Boolean combination of the abstraction predicates, the algorithm is guaranteed to find it.

Consequently, we were able to automatically infer invariants for all the loops for which we initially had to write inductive invariants manually. It should be noted that invariants are restricted to a "weak" language, and that the results discovered by the algorithm are very different than the ones a programmer would write naturally.

The above techniques proved effective for many naturally occurring programs. Successful benchmarks include examples from the TVLA shape analysis framework [42] using singly- and doubly-linked lists and nested lists, as well as some textbook algorithm implementations: several sorting routines and the union-find data structure developed by Tarjan [62].

Chapter 2

Preliminaries

2.1 Hoare-style Verification

— Summary

• Computer programs represent transitions from an input state to an output state • Hoare logic is a formal framework for reasoning about properties • Hoare triples define preconditions and postconditions • Using proof rules, claims written as Hoare triples can be proven •

This section presents a formal proof system for proving properties of imperative programs, based on proof rules for individual language constructs. Historically R.W.Floyd invented rules for reasoning about flow charts [19], and later C.A.R.Hoare modified and extended these to treat programs written as code in an imperative programming language [29]. To demonstrate this approach, consider a simplistic programming language called "While-language" with the following abstract syntax:

$$
\begin{array}{rcl}
a & ::= & n \mid X \mid a + a \mid a - a \mid a \times a \\
b & ::= & \textbf{true} \mid \textbf{false} \mid a = a \mid \neg b \mid b \wedge b \mid b \vee b \\
c & ::= & \mathsf{skip} \mid X := a \mid c; c \mid \mathsf{if}\ b\ \mathsf{then}\ c\ \mathsf{else}\ c \mid \mathsf{while}\ b\ \mathsf{do}\ c
\end{array}
$$

Here, n ranges over integer literals, and X ranges over program variables (occasionally called *locations*). The syntactic group a represents arithmetic expressions; b represents Boolean expressions; and c represents commands. We rely on the reader's intuitive model for understanding the behavior of programs written in While.

Definition 1 (Hoare triple)**.** *A* partial correctness assertion *(also referred to as a* Hoare

$$\{A\}\mathsf{skip}\{A\} \qquad \{Q[a/X]\}X := a\{Q\}$$

$$\frac{\{P\}c_1\{C\} \quad \{C\}c_2\{Q\}}{\{P\}c_1; c_2\{Q\}} \qquad \frac{\{P \wedge [\![B]\!]\}c_1\{Q\} \quad \{P \wedge \neg[\![B]\!]\}c_2\{Q\}}{\{P\}\mathsf{if}\ B\ \mathsf{then}\ c_1\ \mathsf{else}\ c_2\{Q\}}$$

$$\frac{\{P \wedge [\![B]\!]\}c\{P\}}{\{P\}\mathsf{while}\ B\ \mathsf{do}\ c\{P \wedge \neg[\![B]\!]\}} \qquad \frac{\models P \to P' \quad \{P'\}c\{Q'\} \quad \models Q' \to Q}{\{P\}c\{Q\}}$$

Table 2.1: Hoare rules for the basic While-language ([67]).

triple*) has the form*

$$\{P\}\ c\ \{Q\}$$

where c is a command and P, Q are assertions*. We do not formally define the language of assertions here, but assume they are written in some form of logic.*

A partial correctness assertion is said to be valid (written $\models \{P\}\ c\ \{Q\}$) when every successful computation of c from a state satisfying P results in a state satisfying Q.

For example,

$$\models \{X > 3\}\ X := X + 1\ \{X > 4\}$$

Table 2.1 contains a set of inference rules for proving validity of partial correctness assertions. The notation $[\![B]\!]$ is used for the semantics of a Boolean condition, expressed as a logical formula. The proof rules are often called *Hoare rules* and the collection of rules constitutes a proof system called *Hoare logic*.

The last rule (bottom right), called the *consequence rule*, is special because the premises include validity of logical implications. Such implications may be hard to prove for their own worth, and may require, for example, arithmetical reasoning. This is the most complicated rule in the system, and is definitely required; indeed, without it we would not even be able to prove some trivial assertion such as—

$$\{X > 1\}\ \mathsf{skip}\ \{X > 0\}$$

Of course, this makes reasoning as hard as any logical reasoning in the logical language of the assertions.

Proposition 1 (Soundness of Hoare logic)**.** *If an assertion $\{P\}c\{Q\}$ is provable using the proof system of Table 2.1, then $\models \{P\}c\{Q\}$.*

The proof of this proposition is carried out by systematic induction on the proof tree. It is covered in detail in [67].

2.2 Completeness and Weakest-Precondition

— *Summary*

• The weakest (liberal) precondition *wlp* is defined over a program c and a post-condition formula Q • It is the weakest condition that must hold for an input state such that after running c, the output state would satisfy Q • It was shown that *wlp* can be defined recursively for many programming languages • This means that Hoare logic can prove any valid assertion, given a complete proof system for the underlying assertion logic •

As Proposition 1 of the previous section shows, Hoare logic can only prove true assertions. Is the converse also true—that every true assertion is provable in Hoare logic? Clearly, this depends heavily on the nature of the assertion language: the logical language used to describe P and Q. In particular, two important factors come into play:

- Our ability to prove *valid implications* within the assertion logic; this is required by the consequence rule.
- The capacity of the assertion language to express *inductive loop invariants*. This is a subtle point and will be addressed later.

The completeness property of Hoare logic with respect to some assumed completeness of the assertion language is known as *relative completeness*. To show it, we first introduce a new concept.

Definition 2 (Weakest liberal precondition)**.** *The* weakest liberal precondition *of an assertion Q with respect to a command c is defined as the set of states $\sigma \in \Sigma$ where every execution of c starting at state σ either diverges or terminates at a state σ' such that $\sigma' \models Q$.*

The term "liberal" refers to the possibility of divergence (non-termination) of c, and is used to distinguish this term from a more strict variant where termination of c must

$$
\begin{array}{rcl}
wlp[\![\mathsf{skip}]\!](Q) & \stackrel{\text{def}}{=} & Q \\
wlp[\![x := a]\!](Q) & \stackrel{\text{def}}{=} & Q[a/x] \\
wlp[\![c_1 \ ; \ c_2]\!](Q) & \stackrel{\text{def}}{=} & wlp[\![c_1]\!](wlp[\![c_2]\!](Q)) \\
wlp[\![\mathsf{if}\ B\ \mathsf{then}\ c_1\ \mathsf{else}\ c_2]\!](Q) & \stackrel{\text{def}}{=} & [\![B]\!] \wedge wlp[\![c_1]\!](Q) \vee \\
 & & \neg[\![B]\!] \wedge wlp[\![c_2]\!](Q) \\
wlp[\![\mathsf{while}\ B\ \{I\}\ \mathsf{do}\ c]\!](Q) & \stackrel{\text{def}}{=} & I
\end{array}
$$

Table 2.2: Standard rules for computing weakest liberal preconditions for While-language procedures annotated with loop invariants and postconditions. I denotes the loop invariant, $[\![B]\!]$ is the semantics of Boolean program conditions, and Q is the postcondition — all are expressed as first-order formulas.

be preserved. In the remainder of the text, we omit the word "liberal" and just write "weakest precondition", but the intention is always "weakest liberal precondition".

For the claim of relative completeness we would like to show that $wlp[\![c]\!](Q)$ is expressible in the logical language of assertions. If the assertion language is closed under Boolean connectives and syntactic substitution, and if *the command c is loop-free*, then this is certainly the case, and explicit formulas for constructing it are given in the first four lines of Table 2.2. Winskell [67] shows that for such c, it holds that $\models \{P\}c\{Q\}$ if and only if $\models P \rightarrow wlp[\![c]\!]Q$.

Inductive Loop Invariants

In the presence of loops, where the length of the program's execution is not bounded, the situation becomes more difficult, since these programs require reasoning on the sets of *reachable states* from an arbitrary starting state—which may be infinite. This usually requires the use of high-order logic. A common alternative is to require that loops are annotated with an appropriate *inductive loop invariant*.

A *loop invariant* is a condition that holds at the beginning and at the end of every loop iteration. An *inductive loop invariant* is a loop invariant with an additional restriction: satisfaction of the invariant at iteration i of the loop (for $i > 1$) has to follow solely from its satisfaction at iteration $i-1$; it cannot be based on any previous iterations.

To illustrate this subtle, but crucial, difference, consider the example program:

$$X \ := \ 2 \ ; \ \mathsf{while}\ Y > 0\ \mathsf{do}\ (\ X \ := \ 2X - 1 \ ; \ Y \ := \ Y - 1\ )$$

Regardless of the value of Y, it is easy to see that $X > 0$ is a valid loop invariant. X is about to receive the sequence of values 2, 3, 5, 9, ... until at some point the loop terminates. However, it is not an *inductive* invariant: for example, if at iteration $i-1$, X would have the value $\frac{1}{2}$, then at iteration i the value will become 0, violating the condition. The corresponding *inductive* invariant needed to show that X is positive in this case would be $X > 1$ (since $X > 1 \implies 2X - 1 > 1$).

Assume that all loops are annotated with appropriate inductive loop invariants. Then, for each loop, apply the Hoare rule for while:

$$\frac{\{P \wedge [\![B]\!]\}c\{P\}}{\{P\}\mathsf{while}\ B\ \mathsf{do}\ c\{P \wedge \neg[\![B]\!]\}}$$

with P as the inductive loop invariant. This effectively splits the given program into loop-free segments, for which we know Hoare logic to be complete, so we can finish the proof as before.

For example, take the previous program with the following annotations:

$$\{\mathbf{true}\}\ X\ :=\ 2\ ;\ \{X > 1\}\ \mathsf{while}\ Y > 0\ \mathsf{do}\ (\ X\ :=\ 2X - 1\ ;\ Y\ :=\ Y - 1\ )\ \{X > 0\}$$

Here, the pre-condition is **true**, the post-condition is $X > 0$, and the loop invariant is $X > 1$. Thus the proof obligations are

(i) $\quad \{\mathbf{true}\}\ X\ :=\ 2\ \{X > 1\}$

(ii) $\quad \{X > 1 \wedge Y > 0\}\ X\ :=\ 2X - 1\ ;\ Y\ :=\ Y - 1\ \{X > 1\}$

(iii) $\quad \{X > 1 \wedge \neg(Y > 0)\}\ \mathsf{skip}\ \{X > 0\}$

which may be discharged by proving the following implications in the theory of real numbers:

(i) $\quad \mathbf{true} \to wlp[\![X\ :=\ 2]\!](X > 1)$

(ii) $\quad (X > 1 \wedge Y > 0) \to wlp[\![X\ :=\ 2X - 1\ ;\ Y\ :=\ Y - 1]\!](X > 1)$

(iii) $\quad (X > 1 \wedge \neg(Y > 0)) \to wlp[\![\mathsf{skip}]\!](X > 0)$

We can now state a completeness theorem for Hoare logic.

Theorem 1 (Hoare logic—completeness)**.** *Let c be a While-language command where every loop is annotated with a loop invariant. If $\models \{P\}c\{Q\}$ and, furthermore, all loop invariants are* **valid and inductive***, then there exists a Hoare proof for $\{P\}c\{Q\}$.*

2.3 Decidability

— *Summary*

• Transforming a Hoare triple into a formula also provides the other direction • If a logic has a decision procedure for valid formulas, one can also decide the validity of a Hoare triple • Thus effectively checking whether a program meets its specification • In many cases, you can get a counterexample when the specification is violated • Examples of decidable logical languages include Persburger arithmetic, real arithmetic, uninterpreted functions, and EPR •

Generating Verification Conditions Table 2.3 provides the standard rules for computing first-order verification conditions corresponding to Hoare triples using weakest liberal preconditions. An auxiliary function VC_{aux} is used for defining the set of side conditions for the loops occurring in the program. These rules are standard and their soundness and relative completeness have been discussed elsewhere (e.g. see [20]).

The rule for while loop is split into two parts: in the *wlp* we take just the loop invariant, where VC_{aux} asserts that loop invariants are inductive and implies the post-condition for each loop.

Verification conditions allow to completely reduce checking the validity of (annotated) partial correctness assertions to checking validity of logical formulas. To check whether $\models \{P\}c\{Q\}$ as a partial correctness assertion, under the assumption that the inductive loop invariants have been chosen correctly, it is enough to verify that $\models VC_{gen}(\{P\}c\{Q\})$ as a logical statement.

Remark. The rules may generate formulas of exponential size due to the extensive use of syntactic substitution. Another solution can be implemented either by using a set of symbols for every program point, or through the method of Flanagan and Saxe [18]. Such size optimization techniques are not discussed in this thesis.

$VC_{aux}(S, Q) \stackrel{\text{def}}{=} \varnothing$ (for any atomic command S)
$VC_{aux}(S_1; S_2, Q) \stackrel{\text{def}}{=} VC_{aux}(S_1, wlp[\![S_2]\!](Q)) \cup VC_{aux}(S_2, Q)$
$VC_{aux}(\mathsf{if}\ B\ \mathsf{then}\ S_1\ \mathsf{else}\ S_2, Q) \stackrel{\text{def}}{=} VC_{aux}(S_1, Q) \cup VC_{aux}(S_2, Q)$
$VC_{aux}(\mathsf{while}\ B\ \{I\}\ \mathsf{do}\ S, Q) \stackrel{\text{def}}{=} VC_{aux}(S, I)\ \cup$ $\{I \wedge [\![B]\!] \rightarrow wlp[\![S]\!](I), I \wedge \neg[\![B]\!] \rightarrow Q\}$
$VC_{gen}(\{P\}S\{Q\}) \stackrel{\text{def}}{=} \big(P \rightarrow wlp[\![S]\!](Q)\big) \wedge \bigwedge VC_{aux}(S, Q)$

Table 2.3: Standard rules for computing VCs using weakest liberal preconditions for procedures annotated with loop invariants and pre/postconditions. The rules for computing $wlp[\![\ ]\!]$ appear in Table 2.2. The auxiliary function VC_{aux} accumulates a conjunction of VCs for the correctness of loops.

Of course, the problem of checking whether a Hoare triple is valid is undecidable (for example, $\models \{\mathbf{true}\}c\{\mathbf{false}\}$ iff c never terminates), so we can not hope to solve it by reducing it to checking validity of logical formulas. Indeed, checking whether a formula is valid is also undecidable in general; for first-order logic, the problem is recursively enumerable (since one can enumerate all possible proofs and check them), but not recursive. For first-order logic over *finite* structures, the problem is not even r.e., and there is no complete proof system.

However, some subsets of logic are nicely behaved, in the sense that the problem of checking whether a given formula is valid is decidable. Consequently, if one is able to define a language with axiomatic semantics and corresponding $wlp[\![]\!]$ such that the verification conditions are defined in such a logic, then the verification problem for the set of programs expressible in that language becomes decidable by reduction to checking of logical validity.

We give a few examples of decidable logics:

- *Presburger arithmetic* is the logic of natural numbers with only addition. The language contains the constants 0 and 1, and the binary operator $+$, from which all the natural numbers can be constructed. The only relation symbol is $=$ for equality. This is enough to express linear equations over natural numbers, as multiplication by a natural constant can be expressed by successive applications of $+$.

- *Real arithmetic* contains the real constants 0 and 1, equality, an order relation $\leq$, and the operations $+$, $-$, $\cdot$. It was shown by Tarski that the theory of real closed fields can be axiomatized in first-order logic. Furthermore, this theory

admits *quantifier elimination*, so the problem is still decidable if the formulas have quantifiers, even though the complexity becomes non-elementary.

- *Quantifier-free with uninterpreted functions* is the logic where languages are allowed to contain any function and relation symbols, and where these symbols may have any interpretation (in contrast with +, for example, which must be interpreted as addition in all structures); the only designated symbol is = for equality. Formulas must be quantifier-free and contain no variable symbols.

- *Effectively-propositional logic* is a subset of relational logic. It is introduced in the next sub-section.

2.4 Effectively-propositional Logic

— *Summary* ———————————————————————————

• Careful use of quantifiers leads to a decidable fragment of first-order logic • Avoid function symbols and preserve a quantifier prefix of $\exists^*\forall^*$ • For this family, it is known that satisfiablity is reducable to Boolean SAT • Hence, it is also decidable • Despite the exponential complexity, modern solvers solve it efficiently in practice •

———————————————————————————————————

In propositional logic, atomic formulas are just predicate symbols. When viewed as a subset of first-order logic, it means that all predicates are nullary— take no arguments— and therefore there are no terms: no function symbols, no variables, and no quantifiers. The satisfiability problem for propositional formulas is widely known as SAT and is NP-complete.

A slight extension would be to allow arbitrary arity of the predicates, and have nullary function symbols (constants). Formulas are closed by nature and take the form of a Boolean combination of shallow atoms such as:

$$\varphi \;=\; P(a) \rightarrow P(b) \vee Q(c, a)$$

It is easy to convince oneself that if this formula has a model, then it also has one where a, b, and c are all different elements of the domain. This suggest that we

substitute every ground atom with a fresh nullary predicate symbol and get:

$$\varphi' \;=\; P_a \to P_b \vee Q_{ca}$$

Which is equi-satisfiable to the previous one; that is, φ' is satisfiable iff φ is satisfiable. So, we have effectively reduced the satisfiability of ground formulas in this slight extension to regular Boolean SAT.

A more interesting extension involves variables and quantification. Consider a language like the one used to express φ, but with variables and with a quantifier *prefix* (that is, all the quantifiers must occur at the beginning of the formula) of only universal quantifiers.

$$\psi \;=\; \forall\alpha : P(a) \to P(\alpha) \vee Q(c, \alpha)$$

For this type of formula we employ a basic theorem of first-order logic, known as *Herbrand's theorem*.

Definition 3. *Let $\Gamma(\overline{x})$ be a quantifier-free formula over a vocabulary Σ with free variables $\overline{x} = x_1, \ldots, x_k$. A* closed instance *of Γ is any closed formula $\Gamma(\overline{t})$ where $\overline{t} = t_1, \ldots, t_k$ are ground terms over Σ.*

Theorem 2 (Herbrand's Theorem)**.** *Let $\psi = \forall\overline{x} : \Gamma(\overline{x})$. Then ψ is satisfiable iff every finite set of closed instances of Γ is satisfiable as a propositional formula—i.e., with every atomic formula substituted by a distinct predicate symbol (and multiple occurrences of the same atom substituted by the same symbol).*

In the above example of ψ, we have $\Gamma(\alpha) = P(a) \to P(\alpha) \vee Q(c, \alpha)$. Since the vocabulary contains only two constants and no other functions, the closed instances of Γ are only $\Gamma(a)$ and $\Gamma(c)$:

$$P(a) \to P(a) \vee Q(c, a)$$

$$P(a) \to P(c) \vee Q(c, c)$$

Now these are ground formulas similar to φ before, so again these formulas can be reduced to an equi-satisfiable propositional theory:

$$P_a \to P_a \vee Q_{ca}$$

$$P_a \to P_c \vee Q_{cc}$$

Clearly this theory is satisfiable, e.g. by the assignment $\{P_1, P_2, Q_1, Q_2 \mapsto \textbf{false}\}$. This means that ψ is also satisfiable. To construct its model, we take the *Herbrand universe*, which is the set of terms over the vocabulary Σ, in this case $\{a, c\}$, and make it the domain; then we set the truth values of the predicates according to the satisfying Boolean assignment:

P	
a	**false**
c	**false**

Q	a	b
a	X	X
c	**false**	**false**

The X's are "don't-care" values, which may be either **true** or **false**.

Using Herbrand's theorem, the satisfiability of every universal formula over a *relational* vocabulary (with only predicate symbols, constants, and variables) reduces to SAT. In fact, for any formula of the form $\exists\overline{x}\forall\overline{y} : \Gamma(\overline{x}, \overline{y})$ we can apply first-order Skolemization, introducing only new constants, because the existential variables $\overline{x}$ occur only on the outer level and are not nested inside universals. This means that the vocabulary remains relational, and we can carry on with the reduction.

Definition 4 (EPR)**.** *Formulas of the form* $\exists\overline{x}\forall\overline{y} : \Gamma(\overline{x}, \overline{y})$ *are called* Effectively Propositional *formulas.*

The name comes from the fact that they can be simulated, as shown above, by a propositional theory.

We should note that the size of the reduction is clearly exponential. The exponent comes from the number of nested universal quantifiers in the formula, since every universally quantified variable has to be instantiated with all possible constants. Hence, if a large formula can be factored into many small formulas with a small degree of nesting, this exponent can be reduced.

Logic with Equality

The reduction to SAT discusses only uninterpreted relation symbols. It is possible to add a designated equality relation $=$ by the usual technique of encoding the *equality*

axioms:

$$\forall\alpha : \alpha = \alpha \qquad \text{(reflexivity)}$$

$$\forall\alpha, \beta : \alpha = \beta \to \beta = \alpha \qquad \text{(symmetry)}$$

$$\forall\alpha, \beta, \gamma : \alpha = \beta \wedge \beta = \gamma \to \alpha = \gamma \qquad \text{(transitivity)}$$

$$\forall\overline{x}_1, \overline{x}_2 : \overline{x}_1 = \overline{x}_2 \to P(\overline{x}_1) \leftrightarrow P(\overline{x}_2)$$

for every predicate symbol $P \in \Sigma$

where $\overline{x}_1, \overline{x}_2$ are lists of distinct variables with length corresponding to the arity of P. The formula $\overline{x}_1 = \overline{x}_2$ is a shorthand for a conjunction of pointwise equalities.

Since the theory of equality is relational and universal, it can be added to any EPR formula and the reduction explained above would work just the same. The only difference is that to construct a model where the interpretation of $=$ is indeed equality, some elements need to be merged; for example, if we get the following truth table for $=$:

$=$	a	b	c
a	**true**	**false**	**false**
b	**false**	**true**	**true**
c	**false**	**true**	**true**

then the elements b and c should be merged, because $b = c$ holds in the model. The axioms of equality make sure that $=$ is always interpreted as a congruence, and that the merging is always possible because substituting b for c in any predicate results in the same truth value.

Implementation in SAT/SMT Solvers

Seemingly, using the aforementioned reduction to solve the satisfiability problem for EPR requires a preprocessing phase where all ground instances are generated. Since there are exponentially many such instances—where the exponent is a quantifier "depth", that is, how many universal quantifiers are nested—such preprocessing would inevitably require exponential time even in the best case. Modern first-order solvers, however, manage to avoid this thanks to the fact that in many cases, only a small number of these is needed to find a refutation (and so prove that the formula is unsatisfiable). The SMT solver Z3 [12] uses *model-based quantifier instantiation* (MBQI,

explained e.g. in [23]) where variables in quantifiers are instantiated according to partial candidate models for the input formula. This heuristic results in much less than the exponential number of actual ground instances of the input formula, and leads to very good running times in practice.

In the presence of an underlying theory, such as arithmetic or arrays, the power of SMT (satisfiability modulo theory) solvers can be combined to solve even more complicated formulas. Z3 also supports this form of reasoning.

Chapter 3

Pointer Manipulations

This chapter is based on the results published in [36].

3.1 Recursive Data Structures: The Need for Transitive Closure

— Summary

• The concept of pointers in data structures lends itself to first-order axiomatization • A pointer field is thought of as a function mapping one object to another • Problems arise when the two objects are of the same kind • The function can then be applied multiple types • Interesting properties require reasoning about paths between objects, thus naturally requiring the use of transitive closure •

Composite data structures are a desirable feature of a programming language. They allow the programmer to reuse code by combining multiple entities and to define a hierarchy of objects. For example, an object representing a book may contain a reference to an object representing an author.

```
class Author {
  String firstName;
  String lastName;
}

class Book {
```

```
  String title;
  Author author;
}
```

In this example, every `Book` instance is linked to at most one `Author` instance. The heap space may then be described as two disjoint sets of objects, *Book* and *Author*, and a mathematical function $author : Book \rightarrow Author$ representing the pointer field. In this case, if there are no more types defined, then a program having k variables of type `Author` and m variables of type `Book` can manipulate at most $k + 2m$ distinct objects at any given time.

The situation becomes more complicated with the introduction of collections, which require the use of *r*ecursive types. For example, a typical definition of a list looks like the following:

```
class Node {
  String data;
  Node next;
}
```

Fig. 3.1 presents a Java program for *in situ* reversal of a linked list. It is a type-safe version of Fig. 1.1. Every node of the list has a `next` field that points to its successor node in the list. Thus, we can model `next` as a function $next : Node \rightarrow Node$ that maps a node in the list to its successor. Because we are going to use this function a lot in this chapter and the following, we employ the abbreviation n for *next* in order to keep formulas short and readable. Now, assume, for simplicity of the example, that the program *reverse* manipulates the entire heap, that is, the heap consists of just the nodes in the linked list, where the head of the input list is given as a formal parameter h. To describe the heap that is *reachable* from h, we use the formula

$$\forall \alpha : h\langle n^* \rangle \alpha \tag{3.1}$$

Where the special notation $\langle n^* \rangle$ means zero or more applications of n. In this case, since the range of this function is the same as its domain, an **unbounded** number of

```
Node reverse(Node h) {
  0: Node c = h;
  1: Node d = null;
  2: while  3: (c != null) {
       4: Node t = c.next;
       5: c.next = null;
       6: c.next = d;
       7: d = c;
       8: c = t;
  }
  9: return d;
}
```

Figure 3.1: A simple Java program that reverses a list in-place.

$$I_0 \stackrel{\text{def}}{=} ac \wedge \forall\alpha : h\langle n^*\rangle\alpha$$

$$I_3 \stackrel{\text{def}}{=} ac \wedge \forall\alpha, \beta \neq null : \left\{ \begin{array}{ll} \alpha\langle n^*\rangle\beta \Leftrightarrow \beta\langle n_0^*\rangle\alpha & d\langle n^*\rangle\alpha \\ c\langle n^*\rangle\alpha \wedge (\alpha\langle n^*\rangle\beta \Leftrightarrow \alpha\langle n_0^*\rangle\beta) & \neg d\langle n^*\rangle\alpha \end{array} \right\}$$

$$I_9 = ac \wedge \forall\alpha : d\langle n^*\rangle\alpha \wedge (\forall\alpha, \beta : \alpha\langle n^*\rangle\beta \Leftrightarrow \beta\langle n_0^*\rangle\alpha)$$

Table 3.1: AF^R invariants for *reverse* (shown in Fig. 3.1). Note that n, n_0 are function symbols while $\alpha\langle n^*\rangle\beta$, $\alpha\langle n_0^*\rangle\beta$ are atomic propositions on the reachability via directed paths from α to β consisting of n, n_0 edges.

elements can be accessed by repeated application — even for a program with only one variable.

We also assume, for this chapter, that the heap is acyclic, i.e., the formula ac below is a precondition of *reverse*.

$$ac \stackrel{\text{def}}{=} \forall\alpha, \beta : \alpha\langle n^*\rangle\beta \wedge \beta\langle n^*\rangle\alpha \rightarrow \alpha = \beta \tag{3.2}$$

Table 3.1 shows the invariants I_0, I_3 and I_9 that describe a precondition, a loop invariant, and a postcondition of *reverse*. They are expressed in AF^R which permits use of function symbols (e.g. n) in formulas only to express reachability (cf. n^*); moreover, quantifier alternation is not permitted. The notation $\left\{ \begin{array}{ll} f & b \\ g & \neg b \end{array} \right\}$ is shorthand for the conditional $(b \wedge f) \vee (\neg b \wedge g)$.

Note that I_3 and I_9 refer to n_0, the value of n at procedure entry. The postcondition I_9 says that *reverse* preserves acyclicity of the list and updates n so that, upon procedure termination, the links of the original list have been reversed. It also says

that all the nodes are reachable from d in the reversed list. I_3 says that at loop entry c is non-null and moreover, the original list is partially reversed. That is, the part of the list reachable from d has its links reversed wrt. the original list, whereas any node not reachable from d is reachable from c and keeps its original link.

The formulas represented so far are not first-order, due to the use of $\langle n^* \rangle$. This puts these formulas in the class $\mathrm{FO}^{\mathrm{TC}}$ of *f*irst-order logic with transitive closure. This is a very rich logic, which is undecidable [34].

Observe that I_3 and I_9 only refer to n^* and never to n alone. A more natural way to express I_9 would be

$$I_9' \stackrel{\text{def}}{=} ac \wedge \forall \alpha : d\langle n^* \rangle \alpha \wedge (\forall \alpha, \beta : n(\alpha) = \beta \Leftrightarrow n_0(\beta) = \alpha) \tag{3.3}$$

But this mix of n^* and n is risky in this context. We shall immediately see why.

3.2 Deterministic Transitive Closure in FOL

— *Summary*

• It is known that $\mathrm{FO}^{\mathrm{TC}}$ cannot be embedded fully in first-order logic • Some properties of transitive closure may be written as axioms, but the result would never be complete • However, when the transitive closure is of a *f*unction f (in which case it is referred to as *d*eterministic transitive closure, DTC), there is some hope • By carefully expressing properties of f^*, while removing the explicit symbol f from the signature, a precise axiomatization is obtained • Still — it is only complete w.r.t. finite structures, hence a finite-model property is required to ensure correctness •

A natural approach to expressing reachability with the *transitive closure operator*:

Definition 5 (TC). *Let $\varphi(x, x')$ be a first-order formula with two free variables x and x'. We write $(TC_{x,x'}\, \varphi)$ to denote the reflexive, transitive closure of the binary relation expressed by φ. Let FO^{TC} be the logic comprising of all first-order formulas with arbitrary occurrences of TC.*

It is easy to see that $\langle n^* \rangle$ is but a special case of TC:

$$s\langle n^* \rangle t \;\equiv\; \big(\mathrm{TC}_{xy}\, n(x) = y\big)(s, t)$$

$\mathrm{FO}^{\mathrm{TC}}$ is strictly more expressive than first-order logic. As a particular, fundamental example, it is well-known that, according to the compactness theorem of first-order logic, the set $\mathbb{N}$ of natural numbers is not first-order expressible. This means that for every first-order theory whose theorems are valid over $\mathbb{N}$, there must also be non-standard models. Indeed, from the Löwenheim-Skolem theorem, it follows that it would have models of any infinite cardinality. In contrast, the following $\mathrm{FO}^{\mathrm{TC}}$ theory [2]:

$$\begin{aligned}
&\forall x : S(x) \neq 0 \\
&\forall x, y : S(x) = S(y) \rightarrow x = y \\
&\forall x : (\mathrm{TC}_{xy}\, S(x) = y)(0, x) \\
&\forall x : x + 0 = x \\
&\forall x : x + S(y) = S(x + y)
\end{aligned} \tag{3.4}$$

has only models that are isomorphic to $\mathbb{N}$, where S is interpreted as the successor function and $+$ is interpreted as addition. Notice that $(\mathrm{TC}_{xy}\, S(x) = y)$ evaluates to $\leq$, the less-than-or-equals relation. The axiomatization would **not** be complete if we replaced it by the standard equivalent $x \leq y \;\Leftrightarrow\; \exists z : x + z = y$, which is first-order.

It has been shown, however, that adding TC, even with various restrictions of the first-order language, leads immediately to an undecidable logic (that is, the satisfiability/validity check for formulas in this language is undecidable); this can be shown by reducing the satisfiability problem of universal $\mathrm{FO}^{\mathrm{TC}}$ formulas to tiling problems [33].

Attempts have been made to use first-order reasoning to mechanically prove properties of formulas with transitive closure. [44] suggests that we add the axiom

$$T_1[f] \quad \equiv \quad \forall u, v : f_{\mathrm{tc}}(u, v) \leftrightarrow (u = v) \vee \exists w : f(u, w) \wedge f_{\mathrm{tc}}(w, v)$$

Where f_{tc} is a new binary relation symbol used to denote the transitive closure of an existing binary relation whose symbol is f. While sound, it does not suffice to prove many valid $\mathrm{FO}^{\mathrm{TC}}$-theorems, and an induction principle is required, which goes beyond first-order logic. This is not surprising, since $\mathrm{FO}^{\mathrm{TC}}$ could never be fully axiomatized via a first-order theory.

We therefore try to find a fragment of $\mathrm{FO}^{\mathrm{TC}}$ that would be expressive enough to describe interesting properties of pointer data structures, yet not too powerful as to

become undecidable. As a first step to exploring the decidability of formulas containing the transitive closure operator, we define names for several limited classes of formulas.

Definition 6. *Let $t_1, t_2, \ldots t_n$ be logical variables or constant symbols. We define four types of* ***atomic propositions****:*

1. **true** / **false**
2. $t_1 = t_2$ *denoting equality*
3. $R(t_1, t_2, \ldots, t_r)$ *denoting the application of relation symbol R of arity r*
4. $t_1\langle f^*\rangle t_2$ *denoting the existence of $k \geq 0$ such that $f^k(t_1) = t_2$, where $f^0(t_1) \stackrel{\text{def}}{=} t_1$, and $f^{k+1}(t_1) \stackrel{\text{def}}{=} f(f^k(t_1))$*

We say that $t_1\langle f^*\rangle t_2$ is a ***reachability constraint*** *between t_1 and t_2 via the function f.*

- ***Quantifier-free formulas with Reachability*** *(QF^R) are Boolean combinations of such formulas without quantifiers.*
- ***Alternation-free formulas with Reachability*** *(AF^R) are Boolean combinations of such formulas with additional quantifiers of the form $\forall^*{:}\varphi$ or $\exists^*{:}\varphi$ where φ is a QF^R formula.*
- ***Forall-Exists Formulas with Reachability*** *(AE^R) are Boolean combinations of such formulas with additional quantifiers of the form $\forall^*\exists^*{:}\varphi$ where φ is a QF^R formula.*

In particular, $QF^R \subset AF^R \subset AE^R$.

Inverting Reachability Constraints

A crucial step in moving from arbitrary FO^{TC} formulas to AE^R or AF^R formulas is eliminating explicit uses of functions such as the "next" function, n. While this may be difficult for a general graph, we show that it can be done for programs that manipulate singly- and doubly-linked lists. In this section, we informally demonstrate this elimination for acyclic lists. We observe that if n is acyclic, we can construct n^+ from n^* by

$$\alpha\langle n^+\rangle\beta \quad \Leftrightarrow \quad \alpha\langle n^*\rangle\beta \wedge \alpha \neq \beta \tag{3.5}$$

$$\begin{array}{|ll|l} \hline \Gamma_{\text{linOrd}} & \stackrel{\text{def}}{=} & \forall \alpha, \beta : \widehat{n^*}(\alpha, \beta) \wedge \widehat{n^*}(\beta, \alpha) \leftrightarrow \alpha = \beta \quad \wedge \\ & & \forall \alpha, \beta, \gamma : \widehat{n^*}(\alpha, \beta) \wedge \widehat{n^*}(\beta, \gamma) \rightarrow \widehat{n^*}(\alpha, \gamma) \quad \wedge \\ & & \forall \alpha, \beta, \gamma : \widehat{n^*}(\alpha, \beta) \wedge \widehat{n^*}(\alpha, \gamma) \rightarrow (\widehat{n^*}(\beta, \gamma) \vee \widehat{n^*}(\gamma, \beta)) \\ \hline \end{array}$$

Table 3.2: Γ_{linOrd} says all points reachable from a given point are linearly ordered.

Also, since n is a function, the set of nodes reachable from a node α is totally ordered by n^*. Therefore, $n(\alpha)$ is the minimal node in this order that is not α. The minimality is expressed using extra universal quantification in

$$n(\alpha) = \beta \quad \Leftrightarrow \quad \alpha \langle n^+ \rangle \beta \wedge \forall \gamma : \alpha \langle n^+ \rangle \gamma \rightarrow \beta \langle n^* \rangle \gamma \tag{3.6}$$

This "*inversion*" shows that n can be expressed using AF^R formulas. However, caution must be practiced when using the elimination above, because it may introduce unwanted quantifier alternations (see Section 6.1.1). Nevertheless our experiments demonstrate that in a number of commonly occurring examples, the alternation can be removed or otherwise avoided, yielding an equivalent AE^R/AF^R formula.

Decidability of AE^R

Reachability constraints written as $\alpha \langle n^* \rangle \beta$ are not directly expressible in FOL. However, AE^R formulas can be reduced to first-order $\forall^* \exists^*$ formulas without function symbols (which are decidable; see Section 2.4) in the following fashion: Introduce a new binary relation symbol $\widehat{n^*}$ with the intended meaning that $\widehat{n^*}(\alpha, \beta) \leftrightarrow \alpha \langle n^* \rangle \beta$. Even though $\widehat{n^*}$ is an uninterpreted relation, we will consistently maintain the fact that it models reachability. Every formula φ is translated into

$$\varphi' \stackrel{\text{def}}{=} \varphi[\widehat{n^*}(t_1, t_2)/t_1 \langle n^* \rangle t_2] \tag{3.7}$$

For example, the acyclicity relation shown in Eq (3.2) is translated into:

$$\widehat{ac} \stackrel{\text{def}}{=} \forall \alpha, \beta : \widehat{n^*}(\alpha, \beta) \wedge \widehat{n^*}(\beta, \alpha) \rightarrow \alpha = \beta \tag{3.8}$$

We add the consistency rule Γ_{linOrd} shown in Table 3.2, which requires that $\widehat{n^*}$ is a total order. This leads to the following propositions:

Proposition 2 (Simulation of AE^R)**.** *Consider an AE^R formula φ over vocabulary $\mathcal{V} =$*

$\langle \mathcal{C}, \{n\}, \mathcal{R} \rangle$. *Let* $\varphi' \stackrel{\text{def}}{=} \varphi[\widehat{n^*}(t_1, t_2)/t_1\langle n^* \rangle t_2]$. *Then* φ' *is a FO formula over vocabulary* $\mathcal{V}' = \langle \mathcal{C}, \emptyset, \mathcal{R} \cup \{\widehat{n^*}\} \rangle$ *and* φ *is simulated by* $\Gamma_{linOrd} \to \varphi'$ *where* Γ_{linOrd} *is the formula in Table 3.2.*

By "simulate" we mean that the constructed first-order formula is valid iff the given AE^R formula φ is valid. Appendix A.1 contains a proof of this proposition. The proof constructs real models from "simulated" FO models using the reachability inversion (Eq (3.6)).

Finite Model Property One thing to point out regarding the proof of Proposition 2, is that it relies on the finiteness of the domain to reconstruct n from $\widehat{n^*}$. To illustrate, consider the domain Q of rational numbers with the order $\leq$. This interpretation satisfies all the properties due to Γ_{linOrd}, but for every two numbers $x \leq y, x \neq y$, there exists a number z such that $x \leq z \leq y, z \neq x, z \neq y$. In this case no element has a successor, and we cannot use Eq (3.6) to reconstruct a function n such that $n^* = \leq$. This is the main reason for requiring the $\forall^*\exists^*$ quantifier prefix for our formulas; without it, not only does the logic become undecidable, but it is also not complete: not every model of the "simulated" formula has a corresponding model of the original formula in AE^R.

Expressivity of AE^R

Although Æ is a relatively weak logic, it can express interesting properties of lists. Typical predicates that express disjointness of two lists and sharing of tails are expressible in Æ. For example, for two singly-linked lists with headers h, k, $disjoint(h, k) \leftrightarrow \forall \alpha : \alpha \neq null \to \neg(h\langle n^* \rangle \alpha \wedge k\langle n^* \rangle \alpha)$.

Another capability still within the power of Æ is to relax the earlier assumption that the program manipulates the whole memory. We describe a summary of *reverse* on arbitrary acyclic linked lists in a heap that may contain other linked data structures. Realistic programs obey ownership requirements, e.g., the head h of the list *owns* the input list which means that it is impossible to reach one of the list nodes without passing through h. That is,

$$\forall \alpha, \beta : \alpha \neq null \to (h\langle n^* \rangle \alpha \wedge \beta\langle n^* \rangle \alpha) \to h\langle n^* \rangle \beta \tag{3.9}$$

This requirement is conjoined to the precondition, *ac*, of *reverse*. Its postcondition is the conjunction of *ac*, the fact that h_0 and d reach the same nodes, (i.e., $\forall\alpha : h_0\langle n_0^*\rangle\alpha \leftrightarrow d\langle n^*\rangle\alpha$) and

$$\forall\alpha,\beta : \alpha\langle n^*\rangle\beta \leftrightarrow \left\{ \begin{array}{ll} \beta\langle n_0^*\rangle\alpha \wedge \beta \neq null & h_0\langle n_0^*\rangle\alpha \wedge h_0\langle n_0^*\rangle\beta \\ \alpha\langle n_0^*\rangle\beta & \neg h_0\langle n_0^*\rangle\alpha \wedge \neg h_0\langle n_0^*\rangle\beta \\ \textbf{false} & h_0\langle n_0^*\rangle\alpha \wedge \neg h_0\langle n_0^*\rangle\beta \\ \alpha\langle n_0^*\rangle h_0 \wedge \beta = h_0 & \neg h_0\langle n_0^*\rangle\alpha \wedge h_0\langle n_0^*\rangle\beta \end{array} \right\} \tag{3.10}$$

Here, the bracketed formula should be read as a four-way case, i.e., as disjunction of the formulas $h_0\langle n_0^*\rangle\alpha \wedge h_0\langle n_0^*\rangle\beta \wedge \beta\langle n_0^*\rangle\alpha \wedge \beta \neq null$; $\neg h_0\langle n_0^*\rangle\alpha \wedge \neg h_0\langle n_0^*\rangle\beta \wedge \alpha\langle n_0^*\rangle\beta$; $h_0\langle n_0^*\rangle\alpha \wedge \neg h_0\langle n_0^*\rangle\beta \wedge \textbf{false}$; and, $\neg h_0\langle n_0^*\rangle\alpha \wedge h_0\langle n_0^*\rangle\beta \wedge \alpha\langle n_0^*\rangle h_0 \wedge \beta = h_0$. Intuitively, this summary distinguishes between the following four cases: (i) both the source (α) and the target (β) are in the reversed list (ii) both source and target are outside of the reversed list (iii) the source is in the reversed list and the target is not, and (iv) the source is outside and the target is in the reversed list. Cases (i)–(iii) are self-explanatory. For (iv) reachability can occur when there exists a path from α to $h_0 = \beta$. Eq (3.10) is a universal formula with $\langle n_0^*\rangle$, so it is in AF^R. In terms of [56], this means that we assume that the procedure is cutpoint-free, that is, there are no pointers directly into the middle of the list.

The more general case where arbitrary cutpoints occur requires more quantifiers. A non-AF^R formula also arises when we want to specify the behavior of this function in this case (such an attempt is discussed in Section 6.1.2). We defer the issue of cutpoints and handle it extensively in Section 5.

3.3 Updating Deterministic Transitive Closure

— *Summary*

• The problem of *view update* is to maintain some view over a relation, while the underlying relation changes • In context of TC, it means how paths change when edges are added or removed • Recent results in descriptive complexity show that for DTC, the update is expressible in FOL without quantifiers •

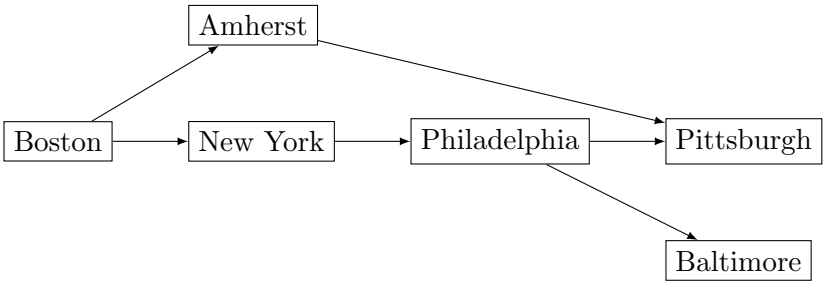

Figure 3.2: Binary relation P as a directed graph.

An interesting development comes from the field of data systems and databases. It is not uncommon for a database application to need the use of transitive closure as part of its functionality. A famous example is the trip planner application: a relation (table) contains available transportation links between major cities or sites:

$E =$

from	**to**
Boston	New York
Boston	Amherst
Amherst	Pittsburgh
New York	Philadelphia
Philadelphia	Pittsburgh
Philadelphia	Baltimore

E is a binary relation; for example, $E(\text{Boston}, \text{Amherst})$ (the column heads are inessential). It can be conceptualized as a directed graph, as shown in Fig. 3.2. One may then be interested in the following query: is there a *p*ath, using edges of E, from Boston to Baltimore? Or, using logical notation, is the value of $E^*(\text{Boston}, \text{Baltimore})$ **true**? Instead of having to scan the graph every time such a query needs to be answered, a database can store an auxiliary table P containing all the pairs $\langle x, y \rangle$ such that $E^*(x, y)$, and then for every incoming query one look-up suffices.

Besides the space trade-off needed to store the table, which may be manageable, there exists the conceptual and computational cost of keeping the auxiliary table P *u*p to date with respect to an ever-changing table E. Changes to E come in the form of two operations: **ins** — insert a new pair, **del** – delete an existing pair.

It has been long known ([32]) that the updates required to $P = E^*$ in order to

compensate for these operations on E, where E is **acyclic** (in the sense that there are no cycles in the graph of Fig. 3.2) can be expressed by first-order formulas:

$$\begin{aligned}
\mathbf{ins}(a,b,E): \quad & P'(x,y) \equiv P(x,y) \vee \big(P(x,a) \wedge P(b,y)\big) \\
\mathbf{del}(a,b,E): \quad & P'(x,y) \equiv P(x,y) \wedge \Big[\neg\big(P(x,a) \wedge P(b,y)\big) \vee \\
& \qquad \exists u,v : \Big(P(x,u) \wedge E(u,v) \wedge P(v,y) \\
& \qquad\quad \wedge P(u,a) \wedge \neg P(v,a) \wedge (a \neq u \vee b \neq v)\Big)\Big]
\end{aligned} \tag{3.11}$$

The first equivalence in Eq (3.11) is very convenient as it describes a quantifier-free correlation between P and P'. The second one is a bit trickier:

- It uses an existential quantifier, which limits the context in which it may be used, when the interest is AE^R formulas.
- It uses both P and E intermixed.

Fortunately, according to recent results in dynamic complexity ([28]), when the relation E is *deterministic*—that is, at every given time, for every element x there is at most one y such that $E(x,y)$— The definition for **del** can be greatly simplified (**ins** is unchanged and is brought here again for completeness):

$$\begin{aligned}
\mathbf{ins}(a,b,E): \quad & P'(x,y) \equiv P(x,y) \vee \big(P(x,a) \wedge P(b,y)\big) \\
\mathbf{del}(a,b,E): \quad & P'(x,y) \equiv P(x,y) \wedge \neg\big(P(x,a) \wedge P(b,y)\big)
\end{aligned} \tag{3.12}$$

3.4 Extending wlp for Pointer Expressions in Linked Lists

— *Summary*

• To handle programs with pointers, we need to address program statements involving pointers • Writes are analogous to edge addition or deletion • Reads are edge traversals • This provides axiomatic semantics for programs based on TC •

In this section we show how to express the weakest liberal preconditions of atomic heap manipulating statements using AE^R formulas, for programs that manipulate

$$
\begin{array}{rcl}
wlp[\![x.n := \text{null}]\!](Q) & \stackrel{\text{def}}{=} & Q[\alpha\langle n^*\rangle\beta \wedge (\neg\alpha\langle n^*\rangle x \vee \beta\langle n^*\rangle x)/\alpha\langle n^*\rangle\beta] \\
wlp[\![x.n := y]\!](Q) & \stackrel{\text{def}}{=} & \neg y\langle n^*\rangle x \wedge \\
 & & Q[\alpha\langle n^*\rangle\beta \vee (\alpha\langle n^*\rangle x \wedge y\langle n^*\rangle\beta)/\alpha\langle n^*\rangle\beta] \\
s\langle n^+\rangle t & \stackrel{\text{def}}{=} & s\langle n^*\rangle t \wedge s \neq t \\
s\langle n\rangle t & \stackrel{\text{def}}{=} & s\langle n^+\rangle t \wedge \forall\gamma : s\langle n^+\rangle\gamma \rightarrow t\langle n^*\rangle\gamma \\
wlp[\![x := y.n]\!](Q) & \stackrel{\text{def}}{=} & \forall\alpha : x\langle n\rangle\alpha \rightarrow Q[\alpha/x] \\
wlp[\![x := \mathsf{new}]\!](Q) & \stackrel{\text{def}}{=} & \forall\alpha : \left(\bigwedge_{p\in Pvar\cup\{null\}} \neg p\langle n^*\rangle\alpha\right) \rightarrow Q[\alpha/x]
\end{array}
$$

Table 3.3: Rules for computing weakest liberal preconditions for an extension of While-language to support heap updates, memory allocation, and pointer dereference.

acyclic singly-linked lists. Table 3.3 shows standard *wlp* computation rules (top part) and the corresponding rules for field update, field read and dynamic allocation (bottom part).

Destructive Update. The correctness of the rule for destructive field update is according to Eq (3.12). The statement $x.n := y$ corresponds to **ins**, and $x.n :=$ null corresponds to **del**. This latter one requires some adaptation because in the update rule for **del** both ends of the deleted edge (a, b) are used, whereas the statement $x.n :=$ null only explicates the source of the edge. Under the assumption that n is deterministic and acyclic, though, the two are equivalent.

Notice the requirement $\neg y\langle n^*\rangle x$ in $wlp[\![x.n := y]\!](Q)$: it makes sure that an edge (x, y) being added does not introduce a cycle into the list. So in fact, the *wlp* asserts both the required post-condition Q **and** the absence of cycles throughout the run of the program.

Field Dereference. The rationale behind the formula for $wlp[\![x := y.n]\!](Q)$ is that if y has a successor, then the formula Q should be satisfied when x is replaced by this successor. The natural way to specify this is using the Hoare assignment rule

$$wlp[\![x := y.f]\!](Q) \stackrel{\text{def}}{=} Q[f(y)/x]$$

However, this rule uses the function n and does not directly express reachability. Instead we will construct a relation r_f such that $r_f(\alpha, \beta) \leftrightarrow f(\alpha) = \beta$ and then use universal

quantifications to "access" the value

$$wlp[\![x := y.f]\!](Q) \stackrel{\text{def}}{=} \forall \alpha : r_f(y, \alpha) \rightarrow Q[\alpha/x]$$

Since, for regular singly-linked lists, n is acyclic, we do not need a symbol for r_n — we can express r_n in terms of n^* as follows. First we observe that $n(\alpha) \neq \alpha$. Also, since n is a function, the set of nodes reachable from α is totally ordered by n^*. Therefore, similarly to Eq (3.6), we can express $r_n(\alpha, \beta)$ as the minimal node β in this order where $\beta \neq \alpha$. Expressing minimality "costs" one extra universal quantification.

In Table 3.3, formula $s\langle n \rangle t$ expresses $r_n(s, t)$ in terms of n^*: $s\langle n \rangle t$ holds if and only if there is a path of length 1 between s and t (source and target). Thus, $y\langle n \rangle \alpha$ is satisfied exactly when $\alpha = n(y)$. If y does not have a successor, then $y\langle n \rangle \alpha$ can only be **true** if $\alpha = \mathit{null}$, hence Q should be satisfied when x is replaced by null, which is in line with the concrete semantics.

Lemma 3 in Appendix A.1 shows that the formula P_n correctly defines n as a relation.

Dynamic allocation. The rule $wlp[\![x := \mathsf{new}]\!](Q)$ expresses the semantic uncertainty caused by the behavior of the memory allocator. We want to be compatible with any run-time memory management, so we do not enforce a concrete allocation policy, but instead require that the allocated node meets some reasonable specifications, namely, that it is different from all values stored in program variables, and that it is unreachable from any other node allocated previously (Note: for programs with explicit `free()`, this assumption relies on the absence of dangling pointers, which can be verified by introducing appropriate assertions; this is, however, beyond the scope of this thesis).

It should be noted that the rules for field dereference and for allocation admit an additional level of quantifier nesting. As explained in Section 2.4, the number of nested universal quantifiers in an EPR formula contributes an exponential component to the complexity of solving it. This is usually not a scalability problem because the program is broken down into its loop-free code segments, which are typically small. If there is a long loop-free block, it can be broken down by (manually) adding assertions in the middle of it.

Closure of AE^R

Notice that Table 2.3 only uses weakest liberal preconditions in a positive context without negations. Therefore, the following proposition (proof in Appendix A.2) holds.

Proposition 3 (VCs in AE^R)**.** *For every program S whose precondition P, postcondition Q, branch conditions, loop conditions, and loop invariants are all expressed as AF^R formulas, $VC_{gen}(\{P\}S\{Q\}) \in AE^R$.*

Optimization remark. The size of the VC can be significantly reduced if instead of syntactic substitution, we introduce a new predicate symbol for each substituted atomic formula, axiomatizing its meaning as a separate formula. For example, $Q[P(\alpha,\beta)/\alpha\langle n^*\rangle\beta]$ (where P is some formula with free variables α, β), can be written more compactly as $Q[r_1(\alpha,\beta)/\alpha\langle n^*\rangle\beta] \wedge \forall\alpha,\beta : r_1(\alpha,\beta) \Leftrightarrow P(\alpha,\beta)$, where r_1 is a fresh predicate symbol. When Q contains many applications of $\langle n^*\rangle$ and P is large, this may save a lot of formula space; roughly, it reduces the order of the VC size from quadratic to linear.

Another way to reduce the VC's size is to define an entirely new set of symbols for each program point. Our original implementation employed this optimization, which is also nice for finding bugs — when the program violates the invariants the SAT solver produces a counterexample with the concrete states at every program point. A similar approach by [18] is also applicable in this case.

3.5 Empirical Results

— *Summary*

• Implemented *wlp*[[]] for While-language with pointers • Proved validity of VC using Z3 as solver for EPR • Our tool was able to verify a collection of naturally-occurring programs •

Details

We have implemented a VC generator, according to Tables Table 3.3 and Table 2.3, in Python, and PLY (Python Lex-Yacc) is employed at the front-end to parse While-language programs annotated with AF^R assertions. The tool verifies that invariants are in the class AF^R and have reachability constraints along a single field (of the form

f^*). The assertions may refer to the store and heap at the entry to the procedure via x_0, f_0, etc. SMT-LIB v2 [5] standard notation is used to format the VC and to invoke Z3. The validity of the VC can be checked by providing its negation to Z3. If Z3 exhibits a satisfying assignment then that serves as counterexample for the correctness of the assertions. If no satisfying assignment exists, then the generated VC is valid, and therefore the program satisfies the assertions.

The output model/counterexample (S-Expression), if one is generated, is then also parsed, so that we have the truth table of n^*. This structure represents the state of the program either at entry or at the beginning of a loop iteration: running the program from this point will violate one or more invariants. To provide feedback to the user, n is recovered by computing Eq (3.6), and then the `pygraphviz` tool is used to visualize and present to the user a directed graph, whose vertices are nodes in the heap, and whose edges are the n pointer fields.

We also implemented two procedures for generating VCs: the first one implements the standard rules shown in Table 2.3 and a second one uses a separate set of (predicate and constant) symbols per program point as a way to reduce the size of the generated VC formula, as explained in the optimization remark above. We only report data on the former since it exhibited shorter running times, through the latter may scale better.

Verification Examples

We have written AF^R loop invariants and procedure pre- and postconditions for 13 example procedures shown in Table 3.5. These are standard benchmarks and what they do can be inferred either from their names or from Table 3.4. We are encouraged by the fact that it was not difficult to express assertions in AF^R for these procedures. The annotated examples and the VC generation tool are publicly available from `bitbucket.org/tausigplan/epr-verification`.

For an example of the annotations used in the benchmarks, see Table 3.1, listing the precondition, loop invariant, and postcondition of *reverse*. More examples are given in Appendix B.

As expected, Z3 is able to verify all the correct programs. Table 3.5 shows statistics for size and complexity of the invariants and the running times for Z3. The tests were conducted on a 1.7GHz Intel Core i5 machine with 4GB of RAM, running OS X 10.7.5. The version of Z3 used was 4.2, compiled for 64-bit Intel architecture (using `gcc` 4.2,

```
Node filter(Node h) {
  Node i = h, j = null;
  while (i != null) {
    if (i!=h && C(i)) j.n := i.n;
    else j := i;
    i := i.n;
  }
  return h;
}
```

Figure 3.3: A simplified Java program that removes elements from a list according to some predicate; for simplicity, we assume that the head is never removed.

LLVM). The solving time reported is wall clock time of the execution of Z3.

To give some account of the programs' sizes, we observe the program summary specification given as pre- and postcondition, count the number of atomic formulas in each of them, and note the depth of quantifier nesting; all our samples had only universal quantifiers. We did the same for each program's loop invariant and for the generated VC_{gen}. Naturally, the size of the VC grows rapidly —approximately at a quadratic rate. This can be observed in the result of the measurements for "SLL: merge", where (i) the size of the invariant and (ii) the number of if-branches and heap manipulating statements, was larger than those in other examples. Still, the time required by Z3 to prove that the VC is valid is short.

For comparison, the size of the formula generated by the alternative implementation, using a separate set of symbols for each program location, was about 10 times shorter — 239 atomic formulas. However, Z3 took a significantly longer time, at 1357ms. We therefore preferred to use the first implementation.

Thanks to the fact that FOL-based tools, and in particular SAT solvers, permit multiple relation symbols we were able to express ordering properties in sorted lists, and so verify order-aware programs such as "insert" and "merge". This situation can be contrasted with tools like Mona ([39],[27]) which are based on monadic second-order logic, where multiple relation symbols are disallowed.

Composing Procedure Summaries

Additionally, we performed experiments in composing summaries of *filter* (Fig. 3.3) and *reverse* (Fig. 3.1). In this case, we wrote the formulas manually and ran Z3 on them, to get a proof of the validity of the equivalences. We found that AF^R-postconditions

of procedure summaries can be sequentially composed for this purpose.

Illustrating $reverse(reverse\ h) = h$. Let n_1^* denote the reachability after running the inner *reverse*, and let n_2^* denote the reachability after running the outer *reverse*. We can express the equivalence of $reverse(reverse\ h)$ and h using the following AF^R implication:

$$\begin{aligned}(\forall \alpha, \beta : \alpha\langle n_1^*\rangle\beta \Leftrightarrow \beta\langle n_0^*\rangle\alpha) \wedge (\forall \alpha, \beta : \alpha\langle n_2^*\rangle\beta \Leftrightarrow \beta\langle n_1^*\rangle\alpha) \\ \Longrightarrow\ \forall \alpha, \beta : \alpha\langle n_2^*\rangle\beta \Leftrightarrow \alpha\langle n_0^*\rangle\beta\end{aligned} \tag{3.13}$$

The second conjunct of the implication's antecedent describes the effect of the inner *reverse* on the initial state while the third conjunct describes the effect of the outer *reverse* on the state resulting from the first. The consequent of the implication states that the initial and final states are equivalent.

Illustrating $filter(C, reverse(h)) = reverse(filter(C, h))$. The program *filter* takes a unary predicate C on nodes, and a list with head h, and returns a list with all nodes satisfying C removed. The postcondition of *filter* is: $\forall \alpha, \beta : \alpha\langle n^*\rangle\beta \Leftrightarrow \neg C(\alpha) \wedge \neg C(\beta) \wedge \alpha\langle n_0^*\rangle\beta$. It says that β is reachable from α in the filtered list provided neither α nor β satisfies C and β was reachable from α initially. We show that the equivalence of $filter(C, reverse(h))$ and $reverse(filter(C, h))$ can be expressed using an AF^R-theorem:

$$\begin{aligned}&(\forall \alpha, \beta : \alpha\langle n_1^*\rangle\beta \Leftrightarrow \beta\langle n_0^*\rangle\alpha) \wedge \\ &(\forall \alpha, \beta : \alpha\langle n_2^*\rangle\beta \Leftrightarrow \neg C(\alpha) \wedge \neg C(\beta) \wedge \alpha\langle n_1^*\rangle\beta) \wedge \\ &(\forall \alpha, \beta : \alpha\langle n_1^{*\prime}\rangle\beta \Leftrightarrow \neg C(\alpha) \wedge \neg C(\beta) \wedge \alpha\langle n_0^*\rangle\beta) \wedge \\ &(\forall \alpha, \beta : \alpha\langle n_2^{*\prime}\rangle\beta \Leftrightarrow \beta\langle n_1^{*\prime}\rangle\alpha) \\ &\Longrightarrow \\ &\forall \alpha, \beta : \alpha\langle n_2^*\rangle\beta \Leftrightarrow \alpha\langle n_2^{*\prime}\rangle\beta\end{aligned}$$

Here n_1^* denotes the reachability after running *reverse* on the input list (first conjunct of the implication's antecedent) and n_2^* denotes the reachability after running *filter* on this reversed list (second conjunct). Similarly $n_1^{*\prime}$ denotes the reachability after running *filter* on the input list (third conjunct) and $n_2^{*\prime}$ denotes the reachability after running

SLL: insert	— Adds a node into a sorted list, preserving order.
SLL: find	— Locates the first item in the list with a given value.
SLL: last	— Returns the last node of the list.
SLL: merge	— Merges two sorted lists into one, preserving order.
SLL: swap	— Exchanges the first and second element of a list.
DLL: fix	— Directs the back-pointer of each node towards the previous node, as required by data structure invariants.
DLL: splice	— Splits a list into two well-formed doubly-linked lists.

Table 3.4: Description of some linked list manipulating programs verified by our tool.

```
Node insert(Node h, Node e) {
  Node i = h, j = null;
  while (i != null && e.val >= i.val) {
    j = i; i = i.n;
  }
  if (j != null) { j.n = e; e.n = i; }
  else { e.n = h; h = e; }
  return h;
}
```

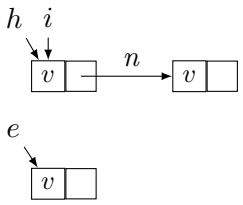

Figure 3.4: Sample counterexample generated for a buggy version of *insert* for a sorted list. Here, the loop invariant required that $\forall\alpha : (h\langle n^*\rangle\alpha \wedge \neg i\langle n^*\rangle\alpha) \rightarrow \alpha <_{val} e$ (where $<_{val}$ is an ordering on nodes according to their values), but the loop condition is **true**, therefore loop will execute one more time, violating this.

reverse on this filtered list (fourth conjunct). The consequent of the implication states that the reachability after performing the first execution path (first *reverse*, then *filter*) is equivalent to that after performing the second execution path (first *filter*, then *reverse*).

Buggy Examples

We also applied the tool to erroneous programs and programs with incorrect assertions. The results, including run-time statistics and formula sizes, are reported in Table 3.6. In addition, we measured the size of the model generated, by observing the size of the generated domain—which reflects the number of nodes in the heap. As expected, Z3 was able to produce a concrete counterexample of a small size. Since these are slight variations of the correct programs, size and running time statistics are similar.

An example of generated output when a program fails to verify can be seen, for the *insert* program, in Fig. 3.4. The tool reports, as part of its output, that a counterexample occurs when $j = null$ and $h.val = i.val = e.val$.

Benchmark	Formula size						Solving time
	P,Q		I		VC		
	#	∀	#	∀	#	∀	(Z3)
SLL: reverse	2	2	11	2	133	3	57ms
SLL: filter	5	1	14	1	280	4	39ms
SLL: create	1	0	1	0	36	3	13ms
SLL: delete	5	0	12	1	152	3	23ms
SLL: deleteAll	3	2	7	2	106	3	32ms
SLL: insert	8	1	6	1	178	3	17ms
SLL: find	7	1	7	1	64	3	15ms
SLL: last	3	0	5	0	74	3	15ms
SLL: merge	14	2	31	2	2255	3	226ms
SLL: rotate	6	1	-	-	73	3	22ms
SLL: swap	14	2	-	-	965	5	26ms
DLL: fix	5	2	11	2	121	3	32ms
DLL: splice	10	2	-	-	167	4	27ms

Table 3.5: Implementation Benchmarks; P,Q — program's specification given as pre- and post-condition, I — loop invariant, VC — verification condition, # — number of atomic formulas, ∀ — quantifier nesting depth

Benchmark	Nature of Defect	Formula size						Solving time	C.e. size
		P,Q		I		VC			
		#	∀	#	∀	#	∀	(Z3)	($\|L\|$)
SLL: find	*null* pointer dereference.	7	1	7	1	64	3	18ms	2
SLL: deleteAll	Loop invariant in annotation is too weak to prove the desired property.	3	2	5	2	68	3	58ms	5
SLL: rotate	Transient cycle introduced during execution.	6	1	-	-	109	3	25ms	3
SLL: insert	Unhandled corner case when an element with the same value already exists in the list — ordering violated.	8	1	6	1	178	3	33ms	4

Table 3.6: Information about benchmarks that demonstrate detection of several kinds of bugs in pointer programs. In addition to the previous measurements, the last column lists the size of the generated counterexample in terms of the number of vertices, or linked-list nodes.

3.6 Related Work for Chapter 3

Decidable Logic. The results in this chapter show that reachability properties of programs manipulating linked lists can be verified using a simple decidable logic AE^R. Many recent decidable logics for reasoning about linked lists have been proposed [7, 46, 48, 70]. In comparison to these works we drastically restrict the way quantifiers are allowed but permit arbitrary use of relations. Thus, strictly speaking our logic is incomparable to the above logics. We show that relations are used even in programs like *reverse* to write procedure summaries such as the one in (3.10) and for expressing numeric orders in sorting programs.

Employing Theorem Provers. The seminal paper on pointer verification [49] provides useful axioms for verifying reachability in linked data structures using theorem provers and conjectures that these axioms are complete for describing reachability. Lev-Ami et al. [44] show that no such axiomatization is possible. This thesis sidesteps the above impossibility results by restricting first-order quantifications and by using the fact that Bernays-Schönfinkel formulas have the finite model property.

Lahiri and Qadeer [40] provide rules for weakest of preconditions for programs with circular linked lists. The formulas are similar to Hesse's [28] but require that the programmer explicitly break the cycle—by annotating the code in a way that designates one edge per cycle as the "closing link". Our framework can be used both with and without the help of the programmer. In practice it may be beneficial to require that the programmer break the cycle in certain cases in order to allow invariants that distinguish between segments in the cycle.

Descriptive Complexity. Descriptive complexity was recently incorporated into the TVLA shape analysis framework [54]. In this chapter we pioneer the use of descriptive complexity for *guaranteeing* that if the programmer writes AF^R assertions and if the program manipulates singly- and doubly-linked lists, then the VCs are guaranteed to be expressible as AE^R formulas.

Chapter 4

Loop Invariants

This chapter is based on the results published in [38].

— *Summary*

• Direct Hoare verification has one critical flaw • A human programmer has to provide loop invariants for the loops occurring in the program • The invariant not only has to hold at every iteration, but also needs to be **inductive** • It means that just by knowing the invariant holds at iteration i, one can prove it would hold at $i+1$ • It is often very difficult to come up with such a suitable loop invariant • Using recent techniques, though, some invariants can be inferred automatically •

As is well known by anyone who ever taught or practiced Hoare logic, an especially intricate part of every Hoare proof, when the program in question involves a loop, is to come up with the *loop invariant*.

The goal of our work is to automatically generate quantified invariants for programs that manipulate singly-linked and doubly-linked list data structures. For a correct program, the invariant generated ensures that the program has no memory-safety violations, such as null-pointer dereferences, and that data structure invariants are preserved. For a program in which it is possible to have a memory-safety violation or for a data-structure invariant to be violated, the algorithm produces a concrete counterexample.

To illustrate the problem and proposed solution, consult the example procedure `insert`, shown in Fig. 4.1, which inserts a new element pointed to by `e` into the non-empty, singly-linked list pointed to by `h`. `insert` is annotated with a pre-condition and

```
void insert(Node e, Node h, Node x) {
  Requires: h ≠ null ∧ h⟨n+⟩x ∧ x⟨n*⟩null ∧ e ≠ null ∧ e⟨n⟩null ∧ ¬h⟨n*⟩e
  Ensures: h ≠ null ∧ h⟨n*⟩e ∧ e⟨n⟩x ∧ x⟨n*⟩null
  Node p = h, q = null;
  while (p != x && p != null) {
    q = p; p = p.n;
  }
  q.n = e; e.n = p;
}
```

Figure 4.1: A procedure to insert the element pointed to by `e` into the non-empty, (unsorted) singly-linked list pointed by `h`, just before the element `x` (which must not be first). The while-loop uses the trailing-pointer idiom: `q` is always one step behind `p`.

a post-condition.

The task of filling in an appropriate loop invariant for `insert` is not trivial because (i) a loop invariant may be more complex than a program's pre-condition or post-condition, and (ii) it is infeasible to enumerate all the potential invariants expressible as CNF formulas over a set of properties. For instance, there are 6 variables in `insert` (including null), so even if we are only interested in properties of the form $x\langle n^* \rangle y$, there are $2^{6\times 6}$ possible clauses that can be created as disjunctions of these atomic formulas. Therefore, the number of candidate invariants that can be formulated with these predicates is $2^{2^{6\times 6}}$. It would be infeasible to investigate them all explicitly.

Intuitively, an invariant for the while-loop in `insert` must assert that q is one step behind p. This is not true, however, for the first iteration where $q = \text{null}$. Consequently, we expect the loop invariant to look something like the following:

$$\begin{aligned} & h \neq \texttt{null} \wedge h\langle n^+ \rangle x \wedge x\langle n^* \rangle \texttt{null} \wedge e \neq \texttt{null} \wedge e\langle n \rangle \texttt{null} \wedge \neg h\langle n^* \rangle e \wedge \\ & (j = \texttt{null} \rightarrow i = h) \wedge \\ & (j \neq \texttt{null} \rightarrow h\langle n^* \rangle k \wedge j\langle n \rangle i) \wedge \\ & i\langle n^* \rangle x \end{aligned} \tag{4.1}$$

The analysis shown here is based on *property-directed reachability* [8]. It starts with the trivial invariant **true**, which is repeatedly refined until it becomes inductive.[1] On each iteration, a concrete counterexample to inductiveness is used to refine the invariant by excluding predicates that are implied by that counterexample.

Although in this chapter we mainly discuss memory-safety properties and data-

[1] An invariant I is inductive at the entry to a loop if whenever the code of the loop body is executed on an arbitrary state that satisfies both I and the loop condition, the result is a state that satisfies I.

structure invariants, the technique can be easily extended to other correctness properties (see Section 4.3).

To the best of our knowledge, our method represents the first shape-analysis algorithm that is capable of (i) reporting concrete counterexamples, or alternatively (ii) establishing that the abstraction in use is not capable of proving the property in question. This result is achieved by combining several existing ideas in a new way:

- The algorithm uses a predicate-abstraction domain [25] in which quantified (closed) formulas express properties of singly and doubly linked lists, and serve as the abstraction predicates. In contrast to most recent work, which uses restricted forms of predicate abstraction — such as Cartesian abstraction [3] — the following analysis uses full predicate abstraction (i.e., the abstraction uses arbitrary Boolean combinations of the predicates).

- The abstraction predicates and language semantics are expressed in the reachability logics defined earlier (Section 3.2), which are decidable using a reduction to SAT.

- The algorithm is property-directed—i.e., its choices are driven by the memory-safety properties to be proven. In particular, the algorithm is based on IC3 [8], which we here refer to as *property-directed reachability* (PDR).

PDR integrates well with full predicate abstraction: in effect, the analysis obtains the same precision as the best abstract transformer for full predicate abstraction, without ever constructing the transformers explicitly. In particular, we cast PDR as a *framework* that is parameterized on

- the logic $\mathcal{L}$ in which the semantics of program statements are expressed, and

- the finite set of predicates that define the abstract domain $\mathcal{A}$ in which invariants can be expressed. An element of $\mathcal{A}$ is an arbitrary Boolean combination of the predicates.

Furthermore, this PDR framework is *relatively complete with respect to the given abstraction*. That is, the analysis is guaranteed to terminate and either (i) verifies the given property, (ii) generates a concrete counterexample to the given property, or (iii) reports that the abstract domain is not expressive enough to establish the proof.

Outcome (ii) is possible because the "frame" structure maintained during PDR can be used to build a trace formula; if the formula is satisfiable, the model can be presented to the user as a concrete counterexample. Moreover, if the analysis fails to prove the property or find a concrete counterexample (outcome (iii)), then there is no way to express an inductive invariant that establishes the property in question using a Boolean combination of the abstraction predicates. Note that outcome (iii) is a much stronger guarantee than what other approaches provide when they terminate with an "unknown" result. For example, abstract interpretation techniques such as [42] may fail to find an invariant even when an appropriate one exists in the given abstract domain.

Key to instantiating the PDR framework for shape analysis was the development of the AF^R and AE^R logics for expressing properties of linked lists in Section 3. AF^R is used to define abstraction predicates, and AE^R is used to express the language semantics. AE^R is a decidable, alternation-free fragment of first-order logic with transitive closure ($\mathrm{FO}^{\mathrm{TC}}$). When applied to list-manipulation programs, atomic formulas of AF^R/AE^R can denote reachability relations between memory locations pointed to by pointer variables, where reachability corresponds to repeated dereferences of pointer fields such as `next` or `prev`. One advantage of AF^R is that it does not require any special-purpose reasoning machinery: an AF^R formula can be converted to a formula in "effectively propositional" logic, which can then be reduced to Boolean SAT solving. That is, in contrast to much previous work on shape analysis, our method makes use of *a general purpose SMT solver*, Z3 [11] (rather than specialized tools developed for reasoning about linked data structures, e.g., [6, 14, 22, 59]).

As we saw, the main restriction in AF^R is that it allows the use of a relation symbol f^* that denotes the transitive closure of a function symbol f, but only limited use of f itself. Despite this restriction, as a language for expressing invariants, AF^R provides a fairly natural abstraction, which means that analysis results should be understandable by non-experts (similar to Fig. 3.4).[2]

This formulation produces a new algorithm for shape analysis that either (i) succeeds, (ii) returns a concrete counterexample, or (iii) returns an abstract trace showing that the abstraction in use is not capable of proving the property in question.

In this chapter of the thesis, the reader will find:

[2] By a "non-expert", we mean someone who has no knowledge of either the analysis algorithm, or the abstraction techniques used inside the algorithm.

- A description of a framework, based on the IC3 algorithm, for finding an inductive invariant in a certain logic fragment (abstract domain) that allows one to prove that a given pre-/post-condition holds or find a concrete counter-example to the property, or, in the case of a negative result, the information that there is no inductive invariant expressible in the abstract domain (Section 4.1).

- An instantiation of the framework for finding invariants of programs that manipulate singly-linked or doubly-linked lists. This instantiation uses AF^R to define a simple predicate-abstraction domain, and is the first application of PDR to establish quantified invariants of programs that manipulate linked lists (Section 4.2).

- An empirical evaluation showing the efficacy of the PDR framework for a set of linked-list programs (Section 4.3).

4.1 Property-Directed Reachability: the IC3 Algorithm for Invariant Inference

— *Summary*

• The IC3 algorithm originates from the field of Model Checking • Originally designed to find invariants in a finite state space, it was shown to explore the state space efficiently • One of its prominent traits is being *p*roperty-directed • It focuses on the property of the reachable state set that is of interest to verify • In cases where the state space is infinite, an abstraction is employed • The result is a full (non-Cartesian) predicate analysis •

In this section, we present an adaptation of the IC3 algorithm ("Incremental Construction of Inductive Clauses for Indubitable Correctness" [8]) that uses predicate abstraction. In this chapter, by *predicate abstraction* we mean the technique that performs verification using a given *fixed* set of abstraction predicates [17], and not techniques that incorporate automatic refinement of the abstraction predicates; e.g. CEGAR. The PDR algorithm shown in Alg. 1 is parameterized by a given finite set of predicates $\mathcal{P}$ expressed in a logic $\mathcal{L}$. The requirements on the logic $\mathcal{L}$ are:

R1 $\mathcal{L}$ is decidable (for satisfiability).

Algorithm 1: $\text{PDR}_{\mathcal{A}}(\mathit{Init}, \rho, \mathit{Bad})$

```
R[−1] := false
R[0] := true
N := 0
while true do
    if there exists 0 ≤ i < N
            such that R[i] = R[i + 1]
    then
        return valid
    end
    (r, A) := Check_A(Bad, R[N])
    if r = unsat then
        N := N + 1
        R[N] := true
    end
    else
        reduce_A(N, A)
    end
end
```

Algorithm 2: $\text{reduce}_{\mathcal{A}}(j, A)$

```
(r, A_1) := Check_A(Init, A)
if r = sat then
    σ := Model(Init ∧ ρ^{N−j} ∧ (Bad)'^{×(N−j)})
    if σ is None then error
    "abstraction failure"
    else error "concrete
    counterexample(σ)"
end
while true do
    (r, A_2) :=
      Check_A((Init)' ∨ (R[j − 1] ∧ ρ), (A)')
    if r = unsat then break
    else reduce_A(j − 1, A_2)
end
for i = 0 . . . j do
    R[i] := R[i] ∧ (¬A_1 ∨ ¬A_2)
end
```

R2 The transition relation for each statement of the programming language can be expressed as a two-vocabulary $\mathcal{L}$ formula.

Then for a particular program, we are given:

- A finite set of predicates $\mathcal{P} = \{p_i \in \mathcal{L}\}, 1 \leq i \leq n$.
- The transition relation of the system as a two-vocabulary formula $\rho \in \mathcal{L}$.
- The initial condition of the system, $\mathit{Init} \in \mathcal{L}$.
- The formula specifying the set of bad states, $\mathit{Bad} \in \mathcal{L}$.

Let $\mathcal{A}$ be the full predicate abstraction domain over the predicates $\mathcal{P}$. That is, each element $A \in \mathcal{A}$ is an *arbitrary* Boolean combination of the predicates $\mathcal{P}$. $A \in \mathcal{A}$ is inductive with respect to Init and ρ if and only if $\mathit{Init} \implies A$ and $A \wedge \rho \implies (A)'$. $(\varphi)'$ renames the vocabulary of constant symbols and relation symbols occurring in φ from $\{c, \ldots, r, \ldots\}$ to $\{c', \ldots, r', \ldots\}$. φ is $(\varphi)'$ stripped of primes.

If the logic $\mathcal{L}$ is propositional logic, then Alg. 1 is an instance of IC3 from [8]. Our presentation is a simplification of more advanced variants [8, 15, 30]. For instance, the presentation omits inductive generalization, although our implementation does implement inductive generalization (see Section 4.3). Furthermore, this simplified presentation brings out the fact that the PDR algorithm is really an analysis *framework* that is parameterized on the set of abstraction predicates $\mathcal{P}$.

The algorithm employs an unbounded array R, where each *frame* $R[i] \in \mathcal{A}$ over-approximates the set of concrete states after executing the loop at most i times. The algorithm maintains an integer N, called the *frame counter*, such that the following invariants hold for all $0 \leq i < N$:

1. *Init* is a subset of all $R[i]$, i.e., $\mathit{Init} \implies R[i]$.
2. The safety requirements are satisfied, i.e., $R[i] \implies \neg \mathit{Bad}$.
3. Each of the $R[i+1]$ includes the states in $R[i]$, i.e., $R[i] \implies R[i+1]$.
4. The successors of $R[i]$ are included in $R[i+1]$, i.e., for all σ, σ' if $\sigma \models R[i]$ and $\langle \sigma, \sigma' \rangle \models \rho$, then $\sigma' \models R[i+1]$.

We illustrate the workings of the algorithm using a simple example, after which we explain the algorithm in detail.

Example 1. *Consider the program* `while (x != y) x = x.n;` *where* `n` *is a pointer field (replaces* `next` *for brevity). Suppose the precondition is* $\mathit{Init} := y \neq \mathit{null} \wedge x\langle n^+ \rangle y$. *We wish to prove the absence of null-dereference; that is,* $\mathit{Bad} := x \neq y \wedge x = \mathit{null}$.

Table 4.1 shows a trace of PDR running with this input; each line represents a SAT query carried out by $\mathrm{PDR}_{\mathcal{A}}$ *(line 1) or by* $\mathrm{reduce}_{\mathcal{A}}$ *(line 2). The predicate abstraction domain* $\mathcal{A}$ *comprises of equalities and* $\langle n^* \rangle$*-paths. At each stage, if the result (r) is "*`unsat`*", then either we unfold one more loop iteration (*$N := N+1$*) or we learn a new clause to add to* $R[j]$ *of the previous step, as marked by the "↗" symbol. If the result is "*`sat`*", the resulting model is used to further refine an earlier clause by recursively calling* $\mathrm{reduce}_{\mathcal{A}}$.

On the first row, we start with $R[0] = \mathbf{true}$, *so definitely* $R[0] \wedge \mathit{Bad}$ *is satisfiable, for example with a model where* $x = y = \mathit{null}$. *The algorithm checks if this model represents a reachable state at iteration 0 (see the second row), and indeed it is not—the result is "*`unsat`*" and the unsat-core is* $x = \mathit{null}$ *(*$\mathit{Init} \wedge x = \mathit{null}$ *is not satisfiable). Therefore, we infer the negation,* $x \neq \mathit{null}$, *and add that to* $R[0]$. *The algorithm progresses in the same manner—e.g., four lines later, we have* $R[0] = (x \neq \mathit{null} \wedge x \neq y)$, *and so on. Eventually, the loop terminates when* $R[i] = R[i+1]$ *for some* i; *in this example, the algorithm terminates because* $R[1] = R[2]$. *The resulting invariant is* $R[2] \equiv (y \neq \mathit{null} \wedge x\langle n^* \rangle y)$, *a slight generalization of Pre in this case.* □

Some terminology used in the PDR algorithm:

N	**Formula**	**Model**	$A := \beta_{\mathcal{A}}(Model)$	**Inferred**
0	$R[0] \wedge Bad$	$(\text{null}, 1)\ 1 \mapsto \text{null}$	$A := x = \text{null} \wedge x \neq y \wedge \neg x\langle n^* \rangle y \wedge y\langle n^* \rangle x$	
0	$((Init)' \vee (R[-1] \wedge \rho)) \wedge (A)'$	**unsat**		$R[0] \models x \neq \text{null}$
0	$R[0] \wedge Bad$	**unsat**		
1	$R[1] \wedge Bad$	$(\text{null}, 1)\ 1 \mapsto \text{null}$	$A := x = \text{null} \wedge x \neq y \wedge \neg x\langle n^* \rangle y \wedge y\langle n^* \rangle x$	
1	$((Init)' \vee (R[0] \wedge \rho)) \wedge (A)'$	$(1, 1)\ 1 \mapsto \text{null}$	$A := x = y \neq \text{null} \wedge x\langle n^* \rangle y \wedge y\langle n^* \rangle x$	
1	$((Init)' \vee (R[-1] \wedge \rho)) \wedge (A)'$	**unsat**		$R[0] \models x \neq y$
1	$R[1] \wedge Bad$	$(\text{null}, 1)\ 1 \mapsto \text{null}$	$A := x = \text{null} \wedge x \neq y \wedge \neg x\langle n^* \rangle y \wedge y\langle n^* \rangle x$	
1	$((Init)' \vee (R[0] \wedge \rho)) \wedge (A)'$	$(1, 2)\ 1, 2 \mapsto \text{null}$	$A := x \neq y \wedge x, y \neq \text{null} \wedge \neg x\langle n^* \rangle y \wedge \neg y\langle n^* \rangle x$	
1	$((Init)' \vee (R[-1] \wedge \rho)) \wedge (A)'$	**unsat**		$R[0] \models x\langle n^* \rangle y$
1	$R[1] \wedge Bad$	$(\text{null}, 1)\ 1 \mapsto \text{null}$	$A := x = \text{null} \wedge x \neq y \wedge \neg x\langle n^* \rangle y \wedge y\langle n^* \rangle x$	
1	$((Init)' \vee (R[0] \wedge \rho)) \wedge (A)'$	**unsat**		$R[1] \models x\langle n^* \rangle y$
1	$R[1] \wedge Bad$	**unsat**		
2	$R[2] \wedge Bad$	$(\text{null}, 1)\ 1 \mapsto \text{null}$	$A := x = \text{null} \wedge x \neq y \wedge \neg x\langle n^* \rangle y \wedge y\langle n^* \rangle x$	
2	$((Init)' \vee (R[1] \wedge \rho)) \wedge (A)'$	**unsat**		$R[2] \models x\langle n^* \rangle y$
2	$R[1] = R[2]$	**valid**		

Table 4.1: Example run with $Init := y \neq \text{null} \wedge x\langle n^+ \rangle y$, $Bad := x \neq y \wedge x = \text{null}$, and $\rho := (x' = n(x))$. Intermediate counterexample models are written as $(x, y)\, E$ where (x, y) is the interpretation of the constant symbols x,y and E are the n-links. The output invariant is $R[1] = R[2] = x\langle n^* \rangle y$.

- `Model`(φ) returns a model σ satisfying φ if it exists, and `None` if it doesn't.
- The abstraction of a model σ, denoted by $\beta_{\mathcal{A}}(\sigma)$, is the cube of predicates from $\mathcal{P}$ that hold in σ: $\beta_{\mathcal{A}}(\sigma) = \bigwedge\{p \mid \sigma \models p, p \in \mathcal{P}\} \wedge \bigwedge\{\neg q \mid \sigma \models \neg q, q \in \mathcal{P}\}$.
- Let $\varphi \in \mathcal{L}$ is a formula in the unprimed vocabulary, $A \in \mathcal{A}$ is a value in the unprimed or primed vocabulary. $Check_{\mathcal{A}}(\varphi, A)$ returns a pair (r, A_1) such that
  - if $\varphi \wedge A$ is satisfiable, then $r =$ `sat` and A_1 is the abstraction of a concrete state in the unprimed vocabulary. That is, if the given A is in the unprimed vocabulary, then $\beta_{\mathcal{A}}(\sigma)$ for some $\sigma \models \varphi \wedge A$; else if A is in the primed vocabulary, then $A_1 = \beta_{\mathcal{A}}(\sigma)$ for some $(\sigma, \sigma') \models \varphi \wedge A$.
  - if $\varphi \wedge A$ is unsatisfiable, then $r =$ `unsat`, and A_1 is a predicate such that $A \implies A_1$ and $\varphi \wedge A_1$ is unsatisfiable. The vocabulary of A_1 is the same as that of A. If A is in the primed vocabulary (as in line 2 of Alg. 2), $Check_{\mathcal{A}}$ drops the primes from A_1 before returning the value.

  A valid choice for A_1 in the unsatisfiable case would be $A_1 = A$ (and indeed the algorithm would still be correct), but ideally A_1 should be the weakest such predicate. For instance, $Check_{\mathcal{A}}(\textbf{false}, A)$ should return (`unsat`, **true**). In practice, when $\varphi \wedge A$ is unsatisfiable, the A_1 returned is an unsat core of $\varphi \wedge A$ constructed

exclusively from conjuncts of A. Such an unsat core is a Boolean combination of predicates in $\mathcal{P}$, and thus is an element of $\mathcal{A}$.

We now give a more detailed explanation of Alg. 1. Each $R[i], i \geq 0$ is initialized to **true** (lines 1 and 1), and $R[-1]$ is **false**. N is initialized to 0 (line 1). At line 1, the algorithm checks whether $R[i] = R[i+1]$ for some $0 \leq i < N$. If true, then an inductive invariant proving unreachability of *Bad* has been found, and the algorithm returns valid (line 1).

At line 1, the algorithm checks whether $R[N] \wedge Bad$ is satisfiable. If it is unsatisfiable, it means that $R[N]$ excludes the states described by *Bad*, and the frame counter N is incremented (line 1). Otherwise, $A \in \mathcal{A}$ represents an abstract state that satisfies $R[N] \wedge Bad$. PDR then attempts to reduce $R[N]$ to try and exclude this abstract counterexample by calling $\text{reduce}_{\mathcal{A}}(N, A)$ (line 1).

The reduce algorithm (Alg. 2) takes as input an integer $j, 0 \leq j \leq N$, and an abstract state $A \in \mathcal{A}$ such that there is a path starting from A of length $N - j$ that reaches *Bad*. Alg. 2 tries to strengthen $R[j]$ so as to exclude A. At line 2, reduce first checks whether $Init \wedge A$ is satisfiable. If it is satisfiable, then there is an abstract trace of length $N - j$ from *Init* to *Bad*, using the transition relation ρ. The call to `Model` at line 2 checks whether there exists a concrete model corresponding to the abstract counterexample. ρ^k denotes k unfoldings of the transition relation ρ; ρ^0 is **true**. $(Bad)'^{\times k}$ denotes k applications of the renaming operation $(\cdot)'$ to *Bad*. If no such concrete model is found, then the abstraction was not precise enough to prove the required property (line 2); otherwise, a concrete counterexample to the property is returned (line 2).

Now consider the case when $Init \wedge A$ is unsatisfiable on line 2. $A_1 \in \mathcal{A}$ returned by the call to $Check_{\mathcal{A}}$ is such that $Init \wedge A_1$ is unsatisfiable; that is, $Init \implies \neg A_1$.

The while-loop on lines 2–2 checks whether the $(N - j)$-length path to *Bad* can be extended backward to an $(N - j + 1)$-length path. In particular, it checks whether $R[j-1] \wedge \rho \wedge (A)'$ is satisfiable. If it is satisfiable, then the algorithm calls reduce recursively on $j - 1$ and A_2 (line 2). If no such backward extension is possible, the algorithm exits the while loop (line 2). Note that if $j = 0$, $Check_{\mathcal{A}}(R[j-1] \wedge \rho, A)$ returns (unsat, **true**), because $R[-1]$ is set to **false**.

The conjunction of $(\neg A_1 \vee \neg A_2)$ to $R[i], 0 \leq i \leq j$, in the loop on lines 2–2 eliminates abstract counterexample A while preserving the required invariants on R. In particular,

the invariant $Init \implies R[i]$ is maintained because $Init \implies \neg A_1$, and hence $Init \implies (R[i] \wedge (\neg A_1 \vee \neg A_2))$. Furthermore, A_2 is the abstract state from which there is a (spurious) path of length $N-j$ to Bad. By the properties of $Check_{\mathcal{A}}$, $\neg A_1$ and $\neg A_2$ are each disjoint from A, and hence $(\neg A_1 \vee \neg A_2)$ is also disjoint from A. Thus, conjoining $(\neg A_1 \vee \neg A_2)$ to $R[i], 0 \leq i \leq j$ eliminates the spurious abstract counterexample A. Lastly, the invariant $R[i] \implies R[i+1]$ is preserved because $(\neg A_1 \vee \neg A_2)$ is conjoined to all $R[i], 0 \leq i \leq j$, and not just $R[j]$.

Formally, the output of $\text{PDR}_{\mathcal{A}}(Init, \rho, Bad)$ is captured by the following theorem:

Theorem 3. *Given (i) the set of abstraction predicates $\mathcal{P} = \{p_i \in \mathcal{L}\}, 1 \leq i \leq n$ where $\mathcal{L}$ is a decidable logic, and the full predicate abstraction domain $\mathcal{A}$ over $\mathcal{P}$, (ii) the initial condition $Init \in \mathcal{L}$, (iii) a transition relation ρ expressed as a two-vocabulary formula in $\mathcal{L}$, and (iv) a formula $Bad \in \mathcal{L}$ specifying the set of bad states, $PDR_{\mathcal{A}}(Init, \rho, Bad)$ terminates, and reports either*

1. valid *if there exists $A \in \mathcal{A}$ s.t. (i) $Init \rightarrow A$, (ii) A is inductive, and (iii) $A \implies \neg Bad$,*
2. *a concrete counterexample trace, which reaches a state satisfying Bad, or*
3. *an abstract trace, if the inductive invariant required to prove the property cannot be expressed as an element of $\mathcal{A}$.* □

The proof of Theorem 3 in Appendix A.3 is based on the observation that, when "abstraction failure" is reported by $\text{reduce}_{\mathcal{A}}(j, A)$, the set of models $\sigma_i \models R[i]$ $(j \leq i < N)$ represents an abstract error trace.

Optimization

There are some optimizations that contribute to reducing the number of iterations required by Alg. 1. We describe one of them in brief and refer to additional material.

Inductive Generalization. Each $R[i]$ is a conjunction of clauses $\varphi_1 \wedge \cdots \wedge \varphi_m$. If we detect that some ψ_j comprising a subset of literals of φ_j, it holds that $R[i] \wedge \rho \wedge \psi_j \models (\psi_j)'$, then ψ_j is *inductive relative to* $R[i]$. In this case, it is safe to conjoin ψ_j to $R[k]$ for $k \leq i+1$. Spurious counter-examples can also be purged if they are inductively blocked. The advantages of this method are explained thoroughly by Bradley [8].

Comparison to Other Predicate-Abstraction Approaches

Our approach differs in two ways from earlier approaches that perform verification via predicate abstraction; that is, verification using a given *fixed* set of abstraction predicates [17]. First, our approach is *property directed*. Our PDR algorithm does not aim to find the strongest invariant expressible in $\mathcal{A}$, instead the algorithm tries to find an invariant that is *strong enough* to verify the given property. Hence, the PDR algorithm could be more efficient in practice, depending on the property to be verified. Second, the PDR algorithm fails to verify the property or provide a concrete counter-example only when it is not possible to express the required inductive invariant as a Boolean combination of the given abstraction predicates $\mathcal{P}$. Most approaches to predicate abstraction use an approximation of the best abstract transformers. As a consequence, earlier approaches might not necessarily find the inductive invariant even if it is expressible in $\mathcal{A}$. On the other hand, approaches that use the best abstract transformers are able to compute the strongest inductive invariant expressible in $\mathcal{A}$ [53, 63, 64, 69], but are not property directed. Furthermore, the frame structure maintained by the PDR algorithm allows it to find concrete counterexamples. In contrast, other approaches to verification via predicate abstraction only return an abstract counterexample trace, from which it is not always possible to construct a concrete counterexample.

4.2 A Useful Predicate Abstraction Domain for Linked Lists

— *Summary*

• We show one possible abstraction space that performs well for linked data structures • The abstraction predicates are expressed as AF^R formulas •

In this section, we describe how $\text{PDR}_{\mathcal{A}}(\mathit{Init}, \rho, \mathit{Bad})$ described in Alg. 1 can be instantiated for verifying linked-list programs. The key insight is the use of the reachability logics for expressing properties of linked lists, as defined in Section 3.2.

We begin by presenting a set of properties that occur naturally when observing programs manipulating linked lists. Table 4.2 contains a list of properties with an English description.

Name	Description	Mnemonic
$x = y$	equality	
$x\langle f\rangle y$	$x\texttt{->}f = y$	
$x\langle f^*\rangle y$	an f path from x to y	
$f.ls\ [x, y]$	unshared f linked-list segment between x and y	
$alloc(x)$	x points to an allocated element	A
$f.stable(h)$	any f-path from h leads to an allocated element	A
$f/b.rev\ [x, y]$	reversed f/b linked-list segment between x and y	R
$f.sorted\ [x, y]$	sorted f list segment between x and y	S

Table 4.2: Predicates for expressing various properties of linked lists whose elements hold data values. x and y denote program variables that point to list elements or null. f and b are parameters that denote pointer fields. (The mnemonics are referred to later in Table 4.5.)

We use two related logics for expressing properties of linked data structures. One of them is AF^R, which already appeared as part of Definition 6. The second is EA^R, in fact the dual of AE^R also defined there. We will need ρ, the transition relation, to be EA^R (as opposed to AE^R) because it is used in a positive context in SAT queries in Alg. 1 and Alg. 2. An alternative set of rules for $wlp[\![]\!]$ that generate EA^R formulas, and are closed under EA^R, is given in Table 4.4 below.

Definition 7. ***(EA^R)*** *(Extension for Definition 6)*

- ***Exists-Forall Formulas with Reachability*** *(EA^R) are Boolean combinations of the atomic propositions with additional quantifiers of the form $\exists^*\forall^*{:}\varphi$ where φ is a QF^R formula.*

In particular, $QF^R \subset AF^R \subset EA^R$. Also, if $\varphi \in AE^R$, then $\neg\varphi$ has an equivalent in EA^R, and vice versa. □

Technically, EA^R forbids *any* use of an individual function symbol f; however, when f defines an acyclic linkage chain—as in acyclic singly linked and doubly linked lists—f can be defined in terms of f^* by using universal quantification to express that an element is the closest in the chain to another element, as demonstrated by Eq (3.6). However, because of the quantifier occurring on the the right-hand side, this macro substitution can only be used in a context that does not introduce a quantifier alternation (e.g., if f occurs in negative context inside a universal quantifier, the resulting formula will not be EA^R).

Name	Formula
$x\langle f\rangle y$	$E_f(x,y)$
$f.ls\ [x,y]$	$\forall\alpha,\beta : x\langle f^*\rangle\alpha \wedge \alpha\langle f^*\rangle y \wedge \beta\langle f^*\rangle\alpha \implies (\beta\langle f^*\rangle x \vee x\langle f^*\rangle\beta)$
$f.stable(h)$	$\forall\alpha : h\langle f^*\rangle\alpha \implies alloc(\alpha)$
$f/b.rev\ [x,y]$	$\forall\alpha,\beta : \left(\begin{array}{l}\alpha \neq \texttt{null} \wedge \beta \neq \texttt{null}\\ \wedge\ x\langle f^*\rangle\alpha \wedge \alpha\langle f^*\rangle y \wedge x\langle f^*\rangle\beta \wedge \beta\langle f^*\rangle y\end{array}\right) \rightarrow \left(\alpha\langle f^*\rangle\beta \leftrightarrow \beta\langle b^*\rangle\alpha\right)$
$f.sorted\ [x,y]$	$\forall\alpha,\beta : \left(\begin{array}{l}\alpha \neq \texttt{null} \wedge \beta \neq \texttt{null}\\ \wedge\ x\langle f^*\rangle\alpha \wedge \alpha\langle f^*\rangle\beta \wedge \beta\langle f^*\rangle y\end{array}\right) \rightarrow \text{dle}(\alpha,\beta)$

Table 4.3: AF^R formulas for the derived predicates shown in Table 4.2. f and b denote pointer fields. dle is an uninterpreted predicate that denotes a total order on the data values. The intention is that $\text{dle}(\alpha,\beta)$ holds whenever $\alpha\texttt{->}d \leq \beta\texttt{->}d$, where d is the data field. We assume that the semantics of dle are enforced by an appropriate total-order background theory.

A Predicate Abstraction Domain that uses AF^R

The abstraction predicates used for verifying properties of linked list programs were introduced informally in Table 4.2. Table 4.3 gives the corresponding formal definition of the predicates as AF^R formulas. Note that all four predicates defined in Table 4.3 are quantified. (The quantified formula for E_f is given in Eq (3.6).) In essence, we use a template-based approach for obtaining quantified invariants: the discovered invariants have a quantifier-free structure, but the atomic formulas can be quantified AF^R formulas. Restricting the formulas in such a manner makes sure that the resulting satisfiability checks in Alg. 2 would be of EA^R formulas.

We now show that the EA^R logic satisfies requirements R1 and R2 for the PDR algorithm stated in Section 4.1.

Decidability of EA^R

To satisfy requirement R1 stated in Section 4.1, we have to show that EA^R is decidable for satisfiability.

EA^R is decidable for satisfiability because any formula in this logic can be translated into the "effectively propositional" decidable logic of $\exists^*\forall^*$ formulas described by Piskac et al. [51]. EA^R includes relations of the form f^* (the reflexive transitive closure of a function symbol f), but only allows limited use of f itself.

Every EA^R formula can be translated into an $\exists^*\forall^*$ formula using the the same steps as in Section 3.2: (i) add a new uninterpreted relation R_f, which is intended to

Command C	$wlp[\![C]\!](Q)$
`assume` φ	$\varphi \rightarrow Q$
`x = y`	$Q[y/x]$
`x = y->f`	$y \neq \texttt{null} \wedge \exists \alpha : (y\langle f \rangle \alpha \wedge Q[\alpha/x])$
`x->f = null`	$x \neq \texttt{null} \wedge Q[\alpha\langle f^* \rangle \beta \wedge (\neg \alpha \langle f^* \rangle x \vee \beta \langle f^* \rangle x)/\alpha \langle f^* \rangle \beta]$
`x->f = y`	$x \neq \texttt{null} \wedge Q[\alpha\langle f^* \rangle \beta \vee (\alpha \langle f^* \rangle x \wedge y \langle f^* \rangle \beta)/\alpha \langle f^* \rangle \beta]$
`x = malloc()`	$\exists \alpha : \neg alloc(\alpha) \wedge Q[(alloc(\beta) \vee (\beta = \alpha \wedge \beta = x))/alloc(\beta))]$
`free(x)`	$alloc(x) \wedge Q[(alloc(\beta) \wedge \beta \neq x)/alloc(\beta))]$

Table 4.4: A revised set of basic $wlp[\![]\!]$ rules for invariant inference. $y\langle f \rangle \alpha$ is the universal formula defined in Eq (3.6). *alloc* stands for a memory location that has been allocated and not subsequently freed.

represent reflexive transitive reachability via f; (ii) add the consistency rule Γ_{linOrd} shown in Table 3.2, which asserts that R_f is a partial order, i.e., reflexive, transitive, acyclic, and linear;[3] and (iii) replace all occurrences of $t_1\langle f^* \rangle t_2$ by $R_f(t_1, t_2)$. (By means of this translation step, acyclicity is built into the logic.)

Proposition 4 (Simulation of EA^R)**.** *Consider EA^R formula φ over vocabulary $\mathcal{V} = \langle \mathcal{C}, \mathcal{F}, \mathcal{R} \rangle$. Let $\varphi' \stackrel{\text{def}}{=} \varphi[R_f(t_1, t_2)/t_1\langle f^* \rangle t_2]$. Then φ' is a first-order formula over vocabulary $\mathcal{V}' = \langle \mathcal{C}, \emptyset, \mathcal{R} \cup \{R_f \colon f \in \mathcal{F}\rangle$, and $\Gamma_{linOrd} \wedge \varphi'$ is satisfiable if and only if the original formula φ is satisfiable.*

This proposition is the dual of Proposition 2 for validity of $\forall^* \exists^*$ formulas.

Axiomatic specification of concrete semantics in EA^R

To satisfy requirement R2 stated in Section 4.1, we have to show that the transition relation for each statement *Cmd* of the programming language can be expressed as a two-vocabulary formula $\rho \in EA^R$, where each program state variable is represented by two constants: c, representing the value of the variable in the pre-state, and c' representing the value of the variable in the post-state. Similarly, each pointer field is represented by two binary relations $f*$ and $f'*$ denoting reachability along f-paths in pre- and post-states, respectively.

Let $wlp[\![Cmd]\!](Q)$ be the weakest liberal precondition of command *Cmd* with respect to $Q \in EA^R$. Then, the transition formula for command *Cmd* is $wlp[\![Cmd]\!](Id)$, where

[3] Note that the order is a partial order and not a total order, because not every pair of elements must be ordered.

Id is a two-vocabulary formula that specifies that the input and the output states are identical, i.e.,

$$Id \stackrel{\text{def}}{=} \bigwedge_{c \in \mathcal{C}} c = c' \wedge \bigwedge_{f \in \mathcal{F}} \forall \alpha, \beta : \alpha\langle f^* \rangle \beta \Leftrightarrow \alpha\langle f'^* \rangle \beta.$$

To show that the concrete semantics of linked list programs can be expressed in EA^R, we have to prove that EA^R is closed under $wlp[\![\,]\!]$; that is, for all commands Cmd and $Q \in EA^R$, $wlp[\![Cmd]\!](Q) \in EA^R$.

Table 4.4 shows rules for computing wlp of atomic commands. This is slightly revised from Table 3.3 which generates $\exists^*\forall^*$ formulas instead of $\forall^*\exists^*$. Note that pointer-related rules in Table 4.4 each include a memory-safety condition to detect `null`-dereferences. For instance, the rule for "`x->f = y`" includes the conjunct "$x \neq$ `null`"; if, in addition, we wish to detect accesses to unallocated memory, the rule would be extended with the conjunct "$alloc(x)$".

The following lemma establishes the soundness and completeness of the wlp rules.

Lemma 1. *Consider a command C of the form defined in Table 4.4 and postcondition Q. Then, $\sigma \models wlp[\![C]\!](Q)$ if and only if the execution of C on σ can yield a state σ' such that $\sigma' \models Q$.*

This lemma is the dual of Theorem 5 (Appendix A.2) for validity of $\forall^*\exists^*$ formulas.

Consider a program with a single loop "`while` *Cond* `do` *Cmd*". Alg. 1 can be used to prove whether or not a precondition $Pre \in AF^R$ before the loop implies that a postcondition $Post \in AF^R$ holds after the loop, if the loop terminates: we supply Alg. 1 with $Init \stackrel{\text{def}}{=} Pre$, $\rho \stackrel{\text{def}}{=} Cond \wedge wlp[\![Cmd]\!](Id)$ and $Bad \stackrel{\text{def}}{=} \neg Cond \wedge \neg Post$. Furthermore, memory safety can be enforced on the loop body by setting $Bad \stackrel{\text{def}}{=} (\neg Cond \wedge \neg Post) \vee (Cond \wedge \neg wlp[\![Cmd]\!](\mathbf{true})$.

4.3 Empirical Results

— *Summary*

• The PDR framework was implemented, parametric of the abstraction predicates • Instantiated with different subsets of predicates • Z3 was again used for generating models and unsat-cores • Tested on a set of programs with linked lists • Invariants were successfully generated for correct programs • We were also able to detect concrete bugs when introduced •

Benchmark	Memory-safety + data-structure integrity					Additional properties				
	$\mathcal{A}$	Time	N	# calls to Z3	# clauses	$\mathcal{A}$	Time	N	# calls to Z3	# clauses
create		1.37	3	28	3		8.19	4	96	7
delete		14.55	4	61	6		9.32	3	67	7
deleteAll	A	6.77	3	72	6	A	37.35	7	308	12
filter		2.37	3	27	4		55.53	5	94	5
insert		26.38	5	220	16		25.25	4	155	13
prev		0.21	2	3	0		11.64	4	118	6
last		0.33	2	3	0		7.49	3	41	4
reverse		5.35	5	128	4		146.42	6	723	11
sorted insert	S	41.07	3	48	7	S	51.46	4	134	10
sorted merge		26.69	4	87	10	S	256.41	5	140	14
make doubly-linked		18.91	3	44	5	R	1086.61	5	112	8

Table 4.5: Experimental results. Column $\mathcal{A}$ signifies the set of predicates used (blank = only the top part of Table 4.2; S = with the addition of the *sorted* predicate family; R = with the addition of the *rev* family; A = with the addition of the *stable* family, where *alloc* conjuncts are added in *wlp* rules). Running time is measured in seconds. N denotes the highest index for a generated element $R[i]$. The number of clauses refers to the inferred loop invariant.

To evaluate the usefulness of the analysis algorithm, we applied it to a collection of sequential procedures that manipulate singly and doubly-linked lists (see Table 4.5). For each program, we report the predicates used, the time (in seconds), the number of PDR frames, the number of calls to Z3, and the size of the resulting inductive invariant, in terms of the number of clauses. All experiments were run on a 1.7GHz Intel Core i5 machine with 4GB of RAM, running OS X 10.7.5. We used version 4.3.2 of Z3 [11], compiled for a 64-bit Intel architecture (using gcc 4.2 and LLVM).

For each of the benchmarks, we verified that the program avoids `null`-dereferences, as well as that it preserves the data-structure invariant that the inputs and outputs are acyclic linked-lists. In addition, for some of the benchmarks we were also able to verify some additional correctness properties. While full functional correctness, or even partial correctness, is hard to achieve using predicate abstraction, we were able to use simple formulas to verify several interesting properties that go beyond memory-safety properties and data-structure invariants. Table 4.6 describes the properties we checked for the various examples. As seen from columns 3, 4, 8, and 9 of the entries for `delete` and `insert` in Table 4.5, trying to prove *stronger* properties can sometimes result in *fewer* iterations being needed, resulting in a *shorter* running time. In the remainder of the examples, handling additional properties beyond memory-safety properties and data-

Benchmark	Property checked
create	Some memory location pointed to by `x` (a global variable) that was allocated prior to the call, is not reachable from the list head, `h`.
delete	The argument `x` is no longer reachable from `h`.
deleteAll	An arbitrary non-null element `x` of the list becomes non-allocated.
filter	Two arbitrary elements `x` and `y` that satisfy the filtering criterion and have an `n`-path between them, maintain that path.
insert	The new element `e` is reachable from `h` and is the direct predecessor of the argument `x`.
last	The function returns the last element of the list.
prev	The function returns the element just before `x`, if one exists.
reverse	If `x` comes before `y` in the input, then `x` should come after `y` in the output.
sorted insert	The list rooted at `h` remains sorted.
make doubly-linked	The resulting `p` is the inverse of `n` within the list rooted at `h`.

Table 4.6: Some correctness properties that can be verified by the analysis procedure. For each of the programs, we have defined suitable *Pre* and *Post* formulas in AF^R.

		Automatic bug finding			
Benchmark	Bug description	Time	N	# calls to Z3	c.e. size
insert	Precondition is too weak (omitted $e \neq$ `null`)	4.46	1	17	8
filter	Potential `null` dereference	6.30	1	21	3
	Typo: list head used instead of list iterator	103.10	3	79	4
reverse	Corrupted data structure: a cycle is created	0.96	1	9	2

Table 4.7: Results of experiments with buggy programs. Running time is measured in seconds. N denotes the highest index for a generated element $R[i]$. "C.e. size" denotes the largest number of individuals in a model in the counterexample trace.

structure invariants required more processing effort, which can be attributed mainly to the larger set of symbols (and hence predicates) in the computation.

Bug Finding

We also ran our analysis on programs containing deliberate bugs, to demonstrate the utility of this approach to bug finding. In all of the cases, the method was able to detect the bug and generate a concrete trace in which the safety or correctness properties are violated. The output in that case is a series of concrete states $\sigma_0..\sigma_N$ where each σ_i contains the set of heap locations, pointer references, and program variables at step i. The experiments and their results are shown in Table 4.7. We found both the length of

the trace and the size of the heap structures to be very small. Their small size makes the traces useful to present to a human programmer, which can help in locating and fixing the bug.

Observations

It is worth noting that for programs where the proof of safety is trivial—because every access is guarded by an appropriate conditional check, such as in `prev` and `last`—the algorithm terminates almost immediately with the correct invariant **true**. This behavior is due to the property-directedness of the approach, in contrast with abstract interpretation, which always tries to find the least fixed point, regardless of the desired property.

We experimented with different refinements of inductive-generalization (Section 4.1). Our algorithm could in many cases succeed even without minimizing the unsat-core, but we observed runs with up to $N = 40$ iterations. On the other hand, the more advanced versions of inductive generalization did not help us: trying to remove literals resulted in a large number of expensive (and useless) solver calls; and blocking spurious counter-examples using inductive generalization also turned out to be quite expensive in our setting.

We also noticed that the analysis procedure is sensitive to the number of abstraction predicates used. In particular, using predicates whose definitions involve quantifiers can affect the running time considerably. When the predicate families $f.sorted\ [x, y]$ and $f/b.rev\ [x, y]$ are added to $\mathcal{A}$, running times can increase substantially (about 20-60 times), as the space of formulas grows much larger. This effect occurred even in the case of `sorted merge`, where we did not attempt to prove an additional correctness property beyond safety and integrity—and indeed there were no occurrences of the added predicates in the loop invariant obtained. As can be seen from Table 4.5, the PDR algorithm *per se* is well-behaved, in the sense that the number of calls to Z3 increased only modestly with the additional predicates. However, each call to Z3 took a lot more time.

In `make doubly-linked`, proving the `prev`/`next`-inverse property requires the predicate family $f/b.rev\ [x, y]$ (see Table 4.3), resulting in a slowdown of more than twenty times over just proving safety and integrity (without the use of the $f/b.rev\ [x, y]$ predicates). For `sorted merge`, we did not prove an additional correctness property beyond safety and

integrity; the numbers reported in columns 6–9 are for proving safety and integrity when the predicate family $f.sorted\ [x, y]$ is also included. These predicates are not needed to prove safety and integrity, nor did they show up in the resulting invariant; however, the analysis time was more than sixty times slower than was needed to prove safety and integrity without the additional predicates, and two orders of magnitude slower than `delete` and `insert`. As can be seen from column 4, about the same number of calls to Z3 were performed for all four examples; however, in the case of (the extended) `sorted merge` and `make doubly-linked`, each call to Z3 took a lot more time.

4.4 Related Work for Chapter 4

The literature on program analysis is vast, and the subject of shape analysis alone has an extensive literature. Thus, in this section we are only able to touch on a few pieces of prior work that relate to the ideas used in this chapter.

Predicate abstraction. Houdini [16] is the first algorithm of which we are aware that aims to identify a loop invariant, given a set of predicates as candidate ingredients. However, Houdini only infers *conjunctive* invariants from a given set of predicates. Santini [64, 65] is a recent algorithm for discovering invariants expressed in terms of a set of candidate predicates. Like our algorithm, Santini is based on full predicate abstraction (i.e., it uses arbitrary Boolean combinations of a set of predicates), and thus is strictly more powerful than Houdini. Santini could make use of the predicates and abstract domain described in this chapter; however, unlike our algorithm, Santini would not be able to report counterexamples when verification fails. Other work infers quantified invariants [31, 61] but does not support the reporting of counterexamples. Templates are used in many tools to define the abstract domains used to represent sets of states, by fixing the form of the constraints permitted. Template Constraint Matrices [60] are based on inequalities in linear real arithmetic (i.e., polyhedra), but leave the linear coefficients as symbolic inputs to the analysis. The values of the coefficients are derived in the course of running the analysis. In comparison, a coefficient in our use of EA^R corresponds to one of the finitely many constants that appear in the program, and we instantiated our templates prior to using PDR.

As mentioned in the beginning of this chapter, PDR meshes well with full predicate abstraction: in effect, the analysis obtains the benefit of the precision of the

abstract transformers for full predicate abstraction, without ever constructing the abstract transformers explicitly. PDR also allows a predicate-abstraction-based tool to create concrete counterexamples when verification fails.

Abstractions based on linked-list segments. Our abstract domain is based on formulas expressed in AF^R, which has very limited capabilities to express properties of stretches of data structures that are not pointed to by a program variable. This feature is similar to the self-imposed limitations on expressibility used in a number of past approaches, including (a) canonical abstraction [59]; (b) a prior method for applying predicate abstraction to linked lists [47]; (c) an abstraction method based on "must-paths" between nodes that are either pointed to by variables or are list-merge points [43]; and (d) domains based on separation logic's list-segment primitive [6, 14] (i.e., "ls$[x, y]$" asserts the existence of a possibly empty list segment running from the node pointed to by x to the node pointed to by y). Decision procedures have been used in previous work to compute the best transformer for individual statements that manipulate linked lists [52, 68].

STRAND and elastic quantified data automata. Recently, Garg et al. developed methods for obtaining quantified invariants for programs that manipulate linked lists via an abstract domain of *quantified data automata* [21, 22]. To create an abstract domain with the right properties, they use a weakened form of automaton—so-called *elastic* quantified data automata—that is unable to observe the details of stretches of data structures that are not pointed to by a program variable. (Thus, an elastic automaton has some of the characteristics of the work based on linked-list segments described above.) An elastic automaton can be converted to a formula in the decidable fragment of the STRAND logic over lists [45].

Other work on IC3/PDR. Our work represents the first application of PDR to programs that manipulate dynamically allocated storage. We chose to use PDR because it has been shown to work extremely well in other domains, such as hardware verification [8, 15]. Subsequently, it was generalized to software model checking for program models that use linear real arithmetic [30] and linear rational arithmetic [9]. Cimatti and Griggio [9] employ a quantifier-elimination procedure for linear rational arithmetic, based on an approximate pre-image operation. Our use of a predicate-abstraction

domain allows us to obtain an approximate pre-image as the unsat core of a single call to an SMT solver (line 2 of Alg. 2).

Chapter 5

Modular Analysis of Procedures

This chapter is based on the results published in [37].

This chapter shows how to harness existing SAT solvers for proving that a program containing (potentially recursive) procedures satisfies its specification and for automatically producing counterexamples when it does not. We concentrate on proving safety properties of imperative programs manipulating linked data structures which is challenging since we need to reason about unbounded memory and destructive pointer updates. The tricky part is to identify a logic which is expressive enough to enable the modular verification of interesting procedures and properties and weak enough to enable sound and complete verification using SAT solvers.

In the previous chapters we have shown how to employ effectively propositional logic for verifying programs manipulating linked lists. However, we see that effectively propositional logic does not suffice to naturally express the effect on the global heap when the local heap of a procedure is accessible via shared nodes from outside. For example, Fig. 5.1 shows a pre- and post-heap before a list pointed-to by h is reversed. The problem is how to express the change in reachability between nodes such as z_i

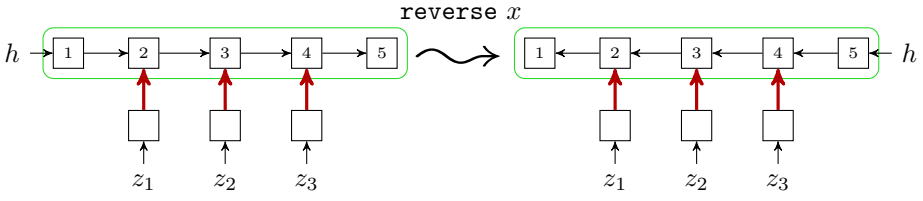

Figure 5.1: Reversing a list pointed to by a head h with many shared nodes accessible from outside the local heap (surrounded by a rounded rectangle).

and list nodes $1, 2, \ldots, 5$: note that, e.g., nodes $3, 4, 5$ are unreachable from z_1 in the post-heap.

This chapter shows that in many cases, including the above example, reachability can still be checked precisely using SAT solvers. Our solution is based on the following principles:

- We follow the standard techniques (e.g., see [4, 41, 66, 71]) by requiring that the programmer define the set of potentially modified elements.
- The programmer only specifies postconditions on local heaps and ignores the effect of paths from the global heap.
- We provide a general and exact *adaptation rule* for adapting postconditions to the global heap. This adaptation rule is expressible in a generalized version of AE^R (Definition 6) called AE^{AR}. AE^{AR} allows an extra function symbol, called the *entry* function, that maps each node u in the global heap into the first node accessible from u in the local heap. In Fig. 5.1, z_1, z_2 and z_3 are mapped to $2, 3$ and 4, respectively. The key facts are that AE^{AR} suffices to precisely define the global reachability relationship after each procedure call and yet any AE^{AR} formula can be simulated by an AE^R formula, and from Proposition 2, by an effectively propositional one. Thus the automatic methods used in Chapter 3 still apply to this significantly more general setting.
- We restrict the verified procedures in order to guarantee that the generated verification condition of every procedure remains in AE^{AR}. The main restrictions are: type correctness, deterministic paths in the heap, limited number of changed list segments in the local heap (each of which may be unbounded) and limited amount of newly created heap sharing by each procedure call. These restrictions are enforced by the generated verification condition in AE^{AR}. This formula is automatically checked by the SAT solver.

Main Results

The results presented in this chapter can be summarized as follows:

- We define a new logic, AE^{AR}, which extends the previously defined AE^R (Definition 6) with a limited idempotent function and yet is equi-satisfiable with EPR.

- We provide a precise adaptation rule in AE^{AR}, expressing the locality property of the change, such that, in conjunction with the formula expressing the post-condition of the local heap, it precisely updates the reachability relation over the global heap.

- We generate a modular verification formula in AE^{AR} for each procedure, asserting that the procedure satisfies its pre- and post-conditions and the above restrictions. This verification condition is sound and complete, i.e., it is valid if and only if the procedure adheres to the restrictions and satisfies its requirements.

- We implemented this tool on top of Z3. We were able to show that many programs can be modularly verified using our methods. They satisfy our restrictions and their invariants can be expressed naturally in AE^{R}.

5.1 The Problem with Global State

5.1.1 A Running Example

To make the discussion more clear, we start with an example program. We use the union-find data structure, due to Tarjan [62], which efficiently maintains a partition of elements, supporting the union of two bins and giving each bin a unique representative. It does so by maintaining a forest using a parent pointer at each node (see Fig. 5.2). [1], Two elements are in the same bin iff they share a common root.

The method `find` requires that the argument x is not null. The formula $\text{tail}_f(x, r_x)$ means that the last node on an f-path from x is r_x, which asserts that the auxiliary variable r_x is equal to the root of x. The procedure changes the pointers of some nodes in the list segment between x and r_x, denoted using the closed interval $[x, r_x]_f$. All the nodes in this segment are set to point directly to r_x.

The return value of `find` (denoted by retval) is r_x. The postcondition uses the symbol $\underline{f}$ denoting the value of f before the method was invoked. Since `find` compresses paths from ancestors of x to single edges to r_x, this root may be shared via new parent pointers. Fig. 5.3 depicts a typical run of `find`.

The method `union` requires that both its arguments are not null. It potentially modifies the ancestors of x and y, i.e., $[x, r_x]_f \cup [y, r_y]_f$. Fig. 5.4 depicts a typical run

[1] We have simplified `union` by not keeping track of the sizes of sets (usually employed in order to attach the smaller set to the larger).

@ requires $x \neq \text{null} \wedge \text{tail}_f(x, r_x)$
@ mod $[x, r_x]_f$
@ ensures $\text{retval} = r_x \wedge$
$\forall \alpha, \beta \in \text{mod} : \alpha\langle f^* \rangle \beta \leftrightarrow \alpha = \beta \vee \beta = r_x$

```
Node find(Node x) {
  Node i = x.f;
  if (i != null) {
    i = find(i);
    x.f = i;
  }
  else {
    i = x;
  }
  return i;
}
```

@ requires $x \neq \text{null} \wedge y \neq \text{null} \wedge \text{tail}_f(x, r_x) \wedge \text{tail}_f(y, r_y)$
@ mod $[x, r_x]_f \cup [y, r_y]_f$
@ ensures

$$\forall \alpha, \beta \in \text{mod} : \begin{array}{l} \big(x\langle \underline{f}^* \rangle \alpha \rightarrow (\alpha\langle f^* \rangle \beta \leftrightarrow \beta = \alpha \vee \beta = r_x \vee \beta = r_y)\big) \\ \wedge \big(y\langle \underline{f}^* \rangle \alpha \rightarrow (\alpha\langle f^* \rangle \beta \leftrightarrow \beta = \alpha \vee \beta = r_y)\big) \end{array}$$

```
void union(Node x, Node y) {
  Node t = find(x);
  Node s = find(y);
  if (t != s) t.f = s;
}
```

Figure 5.2: An annotated implementation of Union-Find in Java. f is the backbone field denoting the parent of a tree node.

of `union`. Notice that the number of cutpoints [56] into the branches of x and y is unbounded.

5.1.2 Working Assumptions

Type correct The procedure manipulates references to dynamically created objects in a type-safe way. For example, we do not support pointer arithmetic.

Deterministic Reachability The specification may use arbitrary uninterpreted relations. It may also use the reachability formula $\alpha\langle f^* \rangle \beta$ meaning that β is reachable from α via zero or more steps along the functional backbone field f. It may not use f in any other way. Until Section 6.2, we require f to be acyclic and we

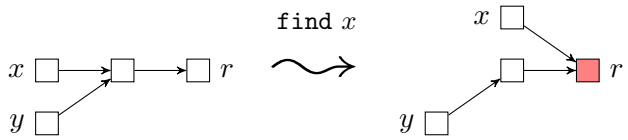

Figure 5.3: An example scenario of running `find` (= return value).

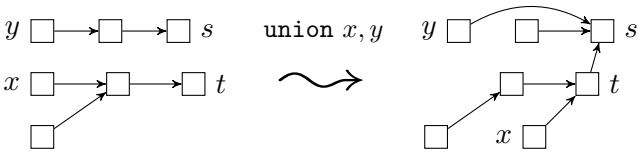

Figure 5.4: An example scenario of running `union`.

restrict our attention to only one backbone field.

Precondition There is a *requires clause* defining the precondition which is written in alternation-free relational first-order logic (AF^R) and may use the relation $f^\star$ (see Definition 6).

Mod-set There is a *modifies clause* defining the mod-set (mod_f), which is the set of potentially changed memory locations. (We include both source and target of every edge that is added or deleted). The modified set may have an unbounded number of vertices, but we require it to be the union of a bounded number of f-intervals, that is chains of vertices through f-pointers.

Postcondition There is an *ensures clause* which exactly defines the new reachability relation $f^\star$ restricted to mod_f. The ensures clause, written in AF^R, may use two vocabularies (employing both $\underline{f}$ and f to refer to the reachability relations before and after.

Bounded new sharing All the new shared nodes — nodes pointed to by more than one node – must be pointed to by local variables at the end of the procedure's execution. This requires that only a bounded number of new shared nodes can be introduced by each procedure call.

Loop-free We assume that all code is loop free, with loops replaced by recursive calls.

Our goal is to reason modularly about a procedure that modifies a subset of the heap. We wish to automatically update the reachability relation in the entire heap

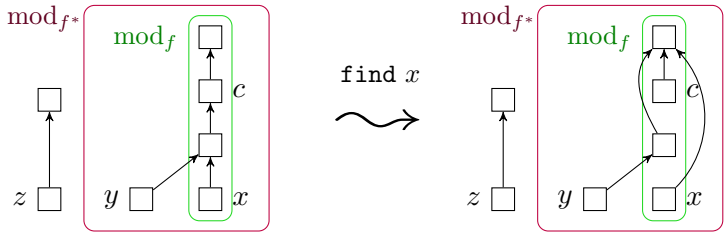

Figure 5.5: A case where changes made by `find` have a non-local effect: $y\langle \underline{f}^* \rangle c$, but $\neg y\langle f^* \rangle c$.

based on the changes to the modified subset. We remark that we are concerned with reachability between any two nodes in the heap, as opposed to only those pointed to by program variables. When we discuss sharing we mean sharing via pointer fields in the heap as opposed to aliasing from stack variables, which does not concern us in this context.

5.1.3 Non-Local Effects

Reachability is inherently non-local: a single edge mutation can affect the reachability of an unbounded number of points that are an unbounded distance from the point of change. Fig. 5.5 contains a typical run of `find`. Two kinds of "frames" are depicted: (i) $\text{mod}_f = [x, r_x)_f$, specified by the programmer, denotes the nodes whose edges can be directly changed by `find`— this is the standard notion of a frame condition; (ii) mod_{f^*} denotes nodes for which f^*, the reachability relation, has changed. We do not and in general we cannot specify this set in a modular way because it usually depends on variables outside the scope of the current function such as y in Fig. 5.5. In the example shown, there is a path from y to c before the call which does not exist after the call. Furthermore, mod_{f^*} can be an arbitrarily large set: in particular, it may not be expressible as the union of a bounded set of intervals: for example, when adding a subtree as a child of some node in another tree, mod_f spans only one edge, whereas mod_{f^*} is the entire subtree added — which may contain an unbounded number of branches.

The postcondition of `find` is sound (every execution of `find` satisfies it), but incomplete: it does not provide a way to determine information concerning paths outside mod, such as from y to c in Fig. 5.5. Therefore, this rule is often not enough to verify the correctness of programs that invoke `find` in larger contexts.

Notice the difficulty of updating the global heap, especially the part $\text{mod}_{f^*} \setminus \text{mod}_f$.

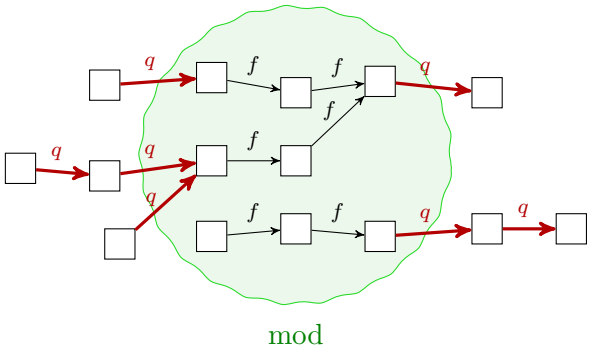

Figure 5.6:
Labeling edges inside and outside mod.

In particular, using only the local specification of `find`, one would not be able to prove that $\neg y\langle f^* \rangle c$. Indeed, the problem is updating the reachability of elements that are **outside** mod; in more complex situations, these elements may be far from the changed interval, and their number may be unbounded.

One possibility to avoid the problem of incompleteness is to specify a postcondition which is specific to the context in which the invocation occurs. However, such a solution requires reasoning per call site and is thus *not modular*. We wish to develop a rule that will fit in all contexts. Reasoning about all contexts is naturally done by quantification.

5.2 An Adaptation Rule for Deterministic Transitive Closure

5.2.1 An FO(TC) Adaptation Rule

A standard way to modularize specifications is to specify the local effect of a procedure and then to use a general adaptation rule (or frame rule) to derive the global effect. In our case, we know that locations outside mod are not modified. Therefore, for example, after a call to `find` a new path from node σ to node τ is either an old path from σ to τ, or it consists of an old path to a node $\alpha \in$ mod, a new path from α to a node $\beta \in$ mod and an old path from β to τ:

$$\begin{aligned} \forall \sigma, \tau : \sigma\langle f^* \rangle \tau \leftrightarrow \sigma\langle q^* \rangle \tau \vee \\ \exists \alpha, \beta \in \text{mod} : \sigma\langle q^* \rangle \alpha \wedge \alpha\langle f^* \rangle \beta \wedge \beta\langle q^* \rangle \tau \end{aligned} \tag{5.1}$$

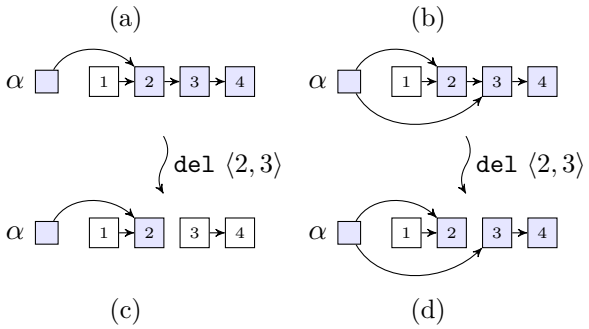

Figure 5.7: Memory states with non-unique pointers where global reasoning about reachability is hard. In the memory state (a), there is one edge from α into the modified-set $\{1, 2, 3, 4\}$, and in memory state (b), there are two edges from α into the same modified-set, $\{1, 2, 3, 4\}$. The two memory states have the same reachability relation and therefore are indistinguishable via reachability alone. The memory states (c) and (d) are obtained from the memory states (a) and (b), respectively, by deleting the edge $\langle 2, 3\rangle$. The reachability in (c) is not the same as in (d), which shows it is impossible to update reachability in general w.r.t. edge deletion, without using the edge relation.

Notice that the condition $\alpha \in \text{mod}$ is expressible as a quantifier-free formula using $\underline{f}^*$ when mod is specified, as it is, as a union of intervals. Here q^* denotes paths through edges, that are definitely unchanged, since at least one of their ends is outside mod (recall that $\underline{f}$ denotes f before the change) —

$$\forall \alpha, \beta : \alpha\langle q\rangle\beta \leftrightarrow \alpha\langle \underline{f}\rangle\beta \wedge (\alpha \notin \text{mod} \vee \beta \notin \text{mod}) \tag{5.2}$$

Thus we distinguish between edges that have both ends in mod, and therefore may have been altered in a manner described by the "ensures" clause, and other edges, which were not changed at all. We therefore define q (and hence, its transitive closure, q^*) to be exactly all the edges "outside" mod, that is, having **at least** one end that does not belong to mod (Eq (5.2)). Now we define all the new paths f^* using q^* and a restriction of f^* to only edges inside mod. See Fig. 5.6 for a depiction of these labels. Every such new path is either constructed entirely of "old" edges q, or is constructed from some path f^* between two nodes $\alpha, \beta \in \text{mod}$, concatenated with some old prefix and some old suffix (each of which may be zero-length). This is expressed by Eq (5.1).

Taking the *adaptation rule* given by Eq (5.1) and Eq (5.2) eliminates superfluous behaviours involving edges outside mod being changed. Notice that these equations are not specific to `find`, but Eq (5.1) does assume that no new path can enter and exit mod multiple times. In this formula α is a placeholder for an *entry-point* into mod and

β is a placeholder for an *exit-point* from mod.

The adaptation rule uses a logic which is too expressive and thus hard for automated reasoning: FO^{TC} is not decidable (in fact, not even recursively enumerable). The first problem is that the q^* relation is not usually first order expressible and generally requires transitive closure. For example, Fig. 5.7 shows that in general the adaptation rule is not necessarily definable using only the reachability relation, when there are multiple outgoing edges per node. We avoid this problem by only reasoning about functional fields, f.

The second problem with Eq (5.1) is that it contains quantifier alternation. α matches an arbitrary node in mod which may be of arbitrary size. Therefore, it is not completely obvious how to avoid existential quantifications.

5.2.2 An Adaptation Rule in a Restricted Logic

We provide an equivalent adaptation rule in a restricted logic, without extra quantifier-alternations. Indeed, our assumptions from Section 5.1.2, especially the deterministic reachability, single backbone field, acyclicity, and the bound on the number of intervals in the mod-set, greatly simplify reasoning about modified paths in the entire heap. The simplification is obtained by employing an additional unary function symbol en^{mod}, where $en^{\text{mod}}(\sigma)$ is meant to denote the entry point of the node σ in mod, that is the *first* node on the (unique) path from σ that enters mod, and null if no such node exists (see Fig. 5.8). Note that since transitive closure is only applied to functions, the entry points such as α in Eq (5.1) is uniquely determined by σ, the origin of the path. A key property of en^{mod} is that on mod itself, en^{mod} acts as identity, and therefore for any $\sigma \in V$ it holds that $en^{\text{mod}}(en^{\text{mod}}(\sigma)) = en^{\text{mod}}(\sigma))$ — that is, the function en^{mod} is *idempotent*. It is important to note that en^{mod} does not change as a result of local modifications in mod. Hence, we do not need to worry about en^{mod} in the pre-state as opposed to the post-state, and we do not need to parametrize en^{mod} by $f/\underline{f}$. Formally, en^{mod} is characterized by the following formula:

$$\begin{aligned} \forall\sigma: \quad & \sigma\langle \underline{f}^* \rangle en^{\text{mod}}(\sigma) \wedge en^{\text{mod}}(\sigma) \in \text{mod} \wedge \\ & \forall\alpha \in \text{mod} : \sigma\langle \underline{f}^* \rangle \alpha \rightarrow en^{\text{mod}}(\sigma)\langle \underline{f}^* \rangle \alpha \end{aligned} \tag{5.3}$$

Using en^{mod} the new adaptation rule adapt[mod] is obtained by considering, for

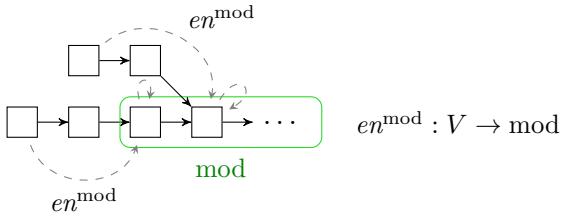

Figure 5.8: The function en^{mod} maps every node σ to the first node in mod reachable from σ. Notice that for any $\alpha \in \text{mod}$, $en^{\text{mod}}(\alpha) = \alpha$ by definition.

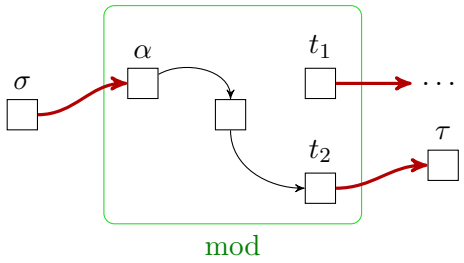

Figure 5.9: This diagram depicts how an arbitrary path from $\sigma \notin \text{mod}$ to $\tau \notin \text{mod}$ is constructed from three segments: $[\sigma, \alpha]_{\underline{f}}$, $[\alpha, t_i]_f$, and $[t_i, \tau]_{\underline{f}}$ (here $i = 2$). Arrows in the diagram denote paths; thick arrows entering and exiting the box denote paths that were not modified since they are outside of mod. Here, $\alpha = en^{\text{mod}}(\sigma)$ is an entry-point and t_1, t_2 are exit-points.

every source and target, the following three cases:

Out-In: The source is out of mod; the target is in;

In-Out: The source is in mod; the target is out;

Out-Out: The source and target are both out of mod.

The full adaptation rule is obtained by taking the conjunction of the formulas for each case (Eq (5.4), Eq (5.5),Eq (5.6)), that are described below, and the formula defining en^{mod} (Eq (5.3)).

Out-In Paths Using en^{mod} we can easily handle paths that *enter* mod. Such paths originate at some $\sigma \notin \text{mod}$ and terminate at some $\tau \in \text{mod}$. Any such path therefore has to go through $en^{\text{mod}}(\sigma)$ as depicted in Fig. 5.9. Thus, the following simple formula can be used:

$$\forall \sigma \notin \text{mod}, \tau \in \text{mod} : \sigma \langle f^* \rangle \tau \leftrightarrow en^{\text{mod}}(\sigma) \langle f^* \rangle \tau \tag{5.4}$$

Observe that for some $\beta \in \text{mod}$, the atomic formula used above, $en^{\text{mod}}(\sigma)\langle f^* \rangle \beta$, corresponds to the FO(TC) sub-formula $\exists \alpha : \sigma \langle q^* \rangle \alpha \wedge \alpha \langle f^* \rangle \beta$ from Eq (5.1).

@ requires $E_f(x, f_x^1) \wedge E_f(f_x^1, f_x^2) \wedge E_f(f_x^3, f_x^2) \wedge$
$x \neq \text{null} \wedge f_x^1 \neq \text{null} \wedge f_x^2 \neq \text{null}$
@ mod $[x, f_x^3]$
@ ensures ...

```
void swap(Node x) {
  Node t = x.f;
  x.f = t.f;
  t.f = x.f.f;
  x.f.f = t;
}
```

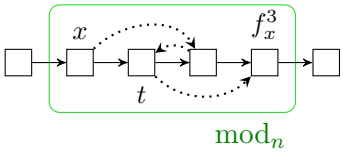

Figure 5.10: A simple function that swaps two adjacent elements following `x` in a singly-linked list. Dotted lines denote the new state after the swap. The notation e.g. $E_f(x, f_x^1)$ denotes the single edge from x to f_x^1 following the f field.

In-Out Paths We now shift attention to paths that *exit* mod. Exit points, that is, last points on some path that belong to mod, are more subtle since you need both ends of the path to determine them. The end of the path is not enough since it can be shared, and the origin of the path is not enough since it can exit the set multiple times, because a path may exit mod and enter it again later. Therefore, we cannot define a function in a similar manner to en^{mod}. The fact that transitive closure is only applied to functions is useful here: every interval $[\alpha, \beta]$ has at most one exit β. We therefore utilize the fact that mod is expressed as a bounded union of intervals — which bounds the potential exit points to a bounded set of terms. We will denote the exit points of mod with t_i.

For example, in the procedure `swap` shown in Fig. 5.10, mod $= [x, f_x^3]$ and there is one exit point $t_1 = f_x^3$ (f_x^3 is a constant set by the precondition to have the value of $f(f(f(x)))$ using the inversion formula Eq (3.6) introduced in Section 3.2.

Observe a general path that originates at some $\sigma \in$ mod and terminates at some $\tau \notin$ mod (see Fig. 5.11). Evidently this path has to go though one of the t_i s, which, as a result of our assumptions, are bounded. Notice that the exit points, too, do not change as a result of modifying edges between nodes in mod. Assume a path from σ to τ and let t_i be the last exit point along that path. This is important, because it lets us know that the segment of the path between t_i and τ comprises solely of unchanged edges — since they are all outside of mod. We can therefore safely use $\underline{f}^*$, rather than q^*, to characterize it. As for the part of the path between σ and t_i, it can be characterized simply by f^*, because σ and t_i are both in mod. Therefore the entire path can be

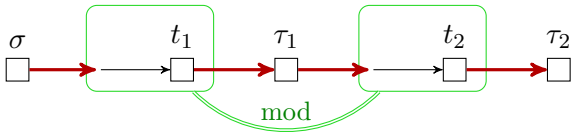

Figure 5.11: A subtle situation occurs when the path from σ passes through multiple exit-points. In such a case, the relevant exit-point for $\sigma\langle f^*\rangle\tau_1$ is t_1, whereas for $\sigma\langle f^*\rangle\tau_2$ and $\tau_1\langle f^*\rangle\tau_2$ it would be t_2.

```
void swap_two(Node a, Node b) {
  swap(a); swap(b);
}
```

Figure 5.12: An example of a procedure where the mod-set is not (essentially) convex.

expressed as $\sigma\langle f^*\rangle t_i \wedge t_i\langle \underline{f}^*\rangle\tau$. Consequently, we obtain the following formula:

$$\begin{aligned}\forall\sigma \in \text{mod}, \tau \notin \text{mod} : \sigma\langle f^*\rangle\tau \leftrightarrow \\ \bigvee_{t_i}(\sigma\langle f^*\rangle t_i \wedge t_i\langle \underline{f}^*\rangle\tau \ \wedge \ \bigwedge_{t_j \neq t_i} t_j \notin [t_i, \tau]_{\underline{f}})\end{aligned} \tag{5.5}$$

Note that Eq (5.5) corresponds to the sub-formula $\exists\beta : \alpha\langle f^*\rangle\beta \wedge \beta\langle q^*\rangle\tau$ in Eq (5.1).

Out-Out Paths For paths between σ and τ, both outside mod, there are two possible situations:

- The path goes **through** mod (as in Fig. 5.9). In this case, we can reuse the in-out case, by taking $en^{\text{mod}}(\sigma)$ instead of σ.
- The path is entirely outside of mod (see Fig. 5.13).

The corresponding formula in this case is:

$$\begin{aligned}\forall\sigma \notin \text{mod}, \tau \notin \text{mod} : \sigma\langle f^*\rangle\tau \leftrightarrow \\ \bigvee_{t_i}(en^{\text{mod}}(\sigma)\langle f^*\rangle t_i \wedge t_i\langle \underline{f}^*\rangle\tau \wedge \ \bigwedge_{t_j \neq t_i} t_j \notin [t_i, \tau]_{\underline{f}}) \vee \\ en^{\text{mod}}(\sigma) = en^{\text{mod}}(\tau) \wedge \sigma\langle \underline{f}^*\rangle\tau\end{aligned} \tag{5.6}$$

Notice that the second disjunct covers both the situation where σ has a path that reaches some node in mod (in which case $en^{\text{mod}}(\sigma) \neq \text{null}$) and the situation where it does not (in which case $en^{\text{mod}}(\sigma) = \text{null}$, so if $\sigma\langle f^*\rangle\tau$, then surely τ cannot reach mod as well, hence $en^{\text{mod}}(\tau) = \text{null}$).

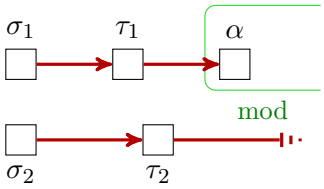

Figure 5.13: Paths that go entirely untouched. $en^{\text{mod}}(\sigma_1) = \alpha$, whereas $en^{\text{mod}}(\sigma_2) =$ null.

To conclude, given mod, adapt[mod] is the conjunction of the three formulas in Eq (5.4), Eq (5.5), Eq (5.6), and the formula defining en^{mod} (Eq (5.3)). To show its adequacy, some more formalism is required, and it is introduced in the following section.

5.2.3 Adaptable Heap Reachability Logic

The utility of en^{mod} in the formulas for adaptation motivates the definition of a logic fragment that would be able to accommodate it.

Definition 8. *The new logic AE^{AR} is obtained by augmenting AE^{R} with unary function symbols, denoted by $g, h_1, \ldots, h_n$ where:*

- *g should be interpreted as an idempotent function: $g\big(g(\alpha)\big) = g(\alpha)$.*
- *The images of $h_1, \ldots, h_n$ are all bounded by some pre-determined parameter N, that is: each h_i takes at most N distinct values.*
- *All terms involving these function symbols have the form $f(z)$, where z is some variable.*

We later show that AE^{AR} suffices for expressing the verification conditions of the programs discussed above. In the typical use case, the function g assigns the entry point in the mod-set for every node (called en^{mod_f} above), and the functions $h_1, \ldots, h_n$ are used for expressing the entry points in inner mod-sets. The main attractive feature of this logic is given in the following theorem.

Theorem 4. *Any AE^{AR}-formula φ can be translated to an equal-valid (first-order) function-free $\forall^*\exists^*$-formula.*

The proof of Theorem 4, given in Appendix A.4, begins by translating φ to a $\forall^*\exists^*$-formula φ' as described in Proposition 2, keeping the function symbols $g, h_1, \ldots, h_n$ as is. These function symbols are then replaced by additional relation and constant symbols,

and extra universal relational formulas are used to enforce the semantic restrictions of them.

5.3 Extending wlp for Procedure Calls

5.3.1 Modular Specifications of Procedure Behaviours

Here we explain how procedure specifications are written in order to support modular verification.

Definition 9. A vocabulary $\mathcal{V} = \langle \mathcal{C}, \{f\}, \mathcal{R} \rangle$ is a triple of constant symbols, a function symbol, and relation symbols. The special constant symbol null is always included. A **state**, M, is a logical structure with universe $|M|$, including null, and $\text{null}^M = \text{null} = f^M(\text{null})$.[2] A state is **appropriate** for an annotated procedure *proc* if its vocabulary includes every symbol occurring in its annotations, and constants corresponding to all of the program variables occurring in *proc*.

The diagrams in this chapter denote program states. For example Fig. 5.1 shows a transition between two states.

Definition 10 (Backbone Differences)**.** *For states $\underline{M}$ and M with the same domain ($|\underline{M}| = |M|$ denotes the domain of M), we denote by $\underline{M}/M$ the structure over the two-vocabulary $\mathcal{V} = \langle \{null\} \cup \mathcal{C} \cup \underline{\mathcal{C}}, \{f, \underline{f}\}, \mathcal{R} \cup \underline{\mathcal{R}} \rangle$ obtained by combining $\underline{M}$ and M in the obvious way. The set $\underline{M} \oplus M$ consists of the "differences between $\underline{M}$ and M excluding null", i.e. $\underline{M} \oplus M = \{u \ : \ f^M(u) \neq f^{\underline{M}}(u)\} \cup \{f^M(u) \ : \ f^M(u) \neq f^{\underline{M}}(u)\} \setminus \{null\}$.*

Modification Set

We must specify the **mod-set**, mod, containing all the endpoints of edges that are modified. Therefore when adding or deleting an edge $\langle s, t \rangle$, both ends — the source, s, and the target, t — are included in mod. Often in programming language semantics, only the source is considered modified. However, thinking of the heap as a graph, it is useful to consider both ends of a modified edge as modified.

Our mod-sets are built from two kinds of intervals:

[2] Remember that null^M is the interpretation of null in the structure M, and f^M is the interpretation of f.

Command	Pre	Mod	Post
`ret = y.f`	$y \neq \text{null} \wedge E_f(y, s)$	$\emptyset$	$ret = s$
`y.f = null`	$y \neq \text{null} \wedge E_f(y, s)$	$[y, s]_f$	$\neg y\langle f^* \rangle s \wedge \neg s\langle f^* \rangle y$
`assume y.f==null;` `y.f = x`	$y \neq \text{null} \wedge E_f(y, \text{null}) \wedge \neg x\langle f^* \rangle y$	$[y, y]_f \cup [x, x]_f$	$y\langle f^* \rangle x \wedge \neg x\langle f^* \rangle y$

Table 5.1: The specifications of atomic commands. s is a local constant denoting the f-field of y. E_f is the inversion formula defined in Eq (3.6).

Definition 11 (Intervals)**.** The *closed interval* $[a, b]_f$ is

$$[a, b]_f \stackrel{\text{def}}{=} \{\alpha \mid a\langle f^* \rangle \alpha \wedge \alpha\langle f^* \rangle b\}$$

and the *half-open interval* $[a, \text{null})_f$ is:

$$[a, \text{null})_f \stackrel{\text{def}}{=} \{\alpha \mid a\langle f^* \rangle \alpha \wedge \alpha\langle f^+ \rangle \text{null}\}$$

(notice that $\alpha\langle f^+ \rangle$null is trivially true in acyclic heaps).

Definition 12 (mod-set)**.** *The mod-set,* mod*, of a procedure is a union* $I_1 \cup I_2 \cup \ldots \cup I_k$*, where each* I_i *may be* $[s_i, t_i]_f$ *or* $[s_i, null)_f$*,* s_i, t_i *are parameters of the procedure or constant symbols occurring in the pre-condition.*

In our examples, the mod-sets of `find` and `union` are written above each procedure, preceded by the symbol "@ mod" (Fig. 5.2). Note that it follows from Definition 12 that "$\alpha \in$ mod" is expressible as a quantifier-free formula.

Definition 13. *Given an appropriate state* M *for proc with modset* mod*,* mod^M *is the set of all elements in* $|M|$ *that are in* mod *(where* mod *is defined by a union of intervals, see Definition 11).*

Pre- and Post-Conditions

The programmer specifies AF^R pre- and post-conditions. Two-vocabulary formulas may be used in the post-conditions where $\underline{f}$ denotes the value of f before the call.

Specifying Atomic Commands

Table 5.1 provides specification of atomic commands. They describe the memory changed by atomic statements and the changes on the local heap.

Accessing a pointer field The statement `ret = y.f` reads the content of the f-field of `y` into `ret`. It requires that `y` is not `null` and that an auxiliary variable `s` point to the f-field of `y` (which may be `null`). It does not modify the heap at all. It sets `ret` to `s`. It is interesting to note that the postcondition is much simpler than the one provided in Section 3.4 because there is no need to specify the effect on the whole heap. In particular, it does not employ quantifiers at all. The quantification is "filled in" by the adaptation rule.

Edge Removals The statement `y.f = null` sets the content of the f-field of `y`, into `null`. It requires that `y` is not `null` and that an auxiliary variable `s` points to the f-field of `y` (which may be `null`). It modifies the node pointed-to by `y` and potentially the node pointed-to by `s`. Notice that the modset includes two elements pointed by y and t, the two end-points of the edge. It removes paths between `y` and `s`. The postcondition assert that there are no paths from y to s. Also, since s is potentially modified, it has to assert that no path is created from s to y.

Edge Additions The statement `y.f = x` is specified assuming without loss of generality, that the statement `y.f = null` was applied before it. Thus, it only handles edge additions. It therefore requires that `y` is not `null` and its f-field is `null`. It modifies the node pointed-to by `y` and potentially the node pointed-to by `x`. It creates a new path from `y` to `x` and asserts the absence of new paths from `x` back to `y`. Again the absence of back paths (denoted by $\neg x\langle f^*\rangle y$) is needed for completeness. The reason is that both the node pointed-to by `x` and `y` are potentially modified. Since `x` is potentially modified, without this assertion, a post-state in which `x.f == y` will be allowed by the postcondition.

5.3.1.1 Soundness and Completeness

We now formalize the notion of soundness and completeness of modular specifications and assert that the specifications of atomic commands are sound and complete.

Definition 14 (Soundness and Completeness of Procedure Specification)**.** *Consider a procedure proc with precondition P, modset* mod*, post-condition Q. We say that $\langle P, \text{mod}, Q\rangle$* ***is sound with respect to*** *proc if for every appropriate pre-state $\underline{M}$ such that $\underline{M} \models P$, and appropriate post-state M which is a potential outcome of the body of proc when executed on $\underline{M}$: (i) $\underline{M} \oplus M \subseteq \text{mod}^{\underline{M}}$, (ii) $\underline{M}/M \models Q$. Such a triple $\langle P, \text{mod}, Q\rangle$* ***is complete with respect to*** *proc if for every appropriate states $\underline{M}, M$ such that (i) $\underline{M} \models P$, (ii) $\underline{M}/M \models Q$, and (iii) $\underline{M} \oplus M \subseteq \text{mod}^{\underline{M}}$, then there exists an execution of the body of proc on $\underline{M}$ whose outcome is M.*

The following proposition establishes the correctness of atomic statements.

Proposition 5 (Soundness and Completeness of Atomic Commands)**.** *The specifications of atomic commands given in Table 5.1 are sound and complete.*

The following lemma establishes the correctness of `find` and `union`, which is interesting since they update an unbounded amount of memory.

Lemma 2 (Soundness and Completeness of Union-Find)**.** *The specification of* ***`find`*** *and* ***`union`*** *in Fig. 5.2 is sound and complete.*

We can now state the following proposition:

Proposition 6 (Soundness and Completeness of adapt[])**.** *Let* mod *be a mod-set of some procedure proc. Let M and $\underline{M}$ be two appropriate states for proc. Then, $\underline{M} \oplus M \subseteq \text{mod}^{\underline{M}}$ iff $\underline{M}/M$ augmented with some interpretation for the function symbol en^{mod} is a model of* adapt[mod]*.*

5.3.2 Generating Verification Condition for Procedure With Sub-calls in AE^{AR}

We will now extend the basic rules for weakest (liberal) precondition with appropriate formulas for procedure call statements.

5.3.2.1 Modular Verification Conditions

The modular verification condition will also contain a conjunct for checking that mod affected by the invoked procedure is a subset of the "outer" mod. This way the specified restriction can be checked in AE^{AR} and the SMT solver can therefore be used to enforce it automatically.

@ requires P_{proc}
@ mod mod_{proc}
@ ensures Q_{proc}
return-type $proc(\overline{x})$ { ... }

Figure 5.14: Specification of *proc* with placeholders.

$$
\begin{aligned}
wlp[\![r := proc(\overline{a})]\!](Q) \stackrel{\text{def}}{=}\ & \\
& P_{proc}[\overline{a}/\overline{x}] \wedge \\
& \forall \alpha : \alpha \in \text{mod}_{proc}[\overline{a}/\overline{x}] \rightarrow \alpha \in \text{mod}_{prog} \wedge \\
& \forall \zeta : Q_{proc}[\overline{a}/\overline{x}, \widehat{f}/f, f/\underline{f}, \zeta/\text{retval}] \wedge \\
& \quad \text{adapt}\left[\text{mod}_{proc}[f/\underline{f}]\right][\overline{a}/\overline{x}, \widehat{f}/f, f/\underline{f}] \rightarrow \\
& \quad Q[\widehat{f}/f, \zeta/r]
\end{aligned}
$$

Table 5.2: Computing the weakest (liberal) precondition for a statement containing a procedure call. r is a local variable that is assigned the return value; $\overline{a}$ are the actual arguments passed. $\widehat{f}$ is a fresh function symbol.

5.3.2.2 Weakest-precondition of Call Statements

As discussed in Section 5.1.2, the specification as it appears in the "ensures" clause of a procedure's contract is a local one, and in order to make reasoning complete we need to adapt it in order to handle arbitrary contexts. This is done by conjoining Q_{proc} occurring in the "ensures" clause from the specification of *proc* with the universal adaptation rule adapt[mod], where mod is replaced with the mod-set as specified in the "modifies" clause of *proc*.

Table 5.2 presents a formula for the weakest-precondition of a statement containing the single procedure call, where the invoked procedure has the specifications as in Fig. 5.14, where "*proc*" has the formal parameters $\overline{x} = x_1, \ldots, x_k$, and it is used with $\overline{a} = a_1, \ldots, a_k$ (used in the formula) as the actual arguments for a specific procedure call; we assume w.l.g. that each a_i is a local variable of the calling procedure.

In general it is not obvious how to enforce that the set of locations modified by inner calls is a subset of the set of locations declared by the outer procedure. Moreover, this can be tricky to check since it depends on aliasing and paths between nodes pointed to by different variables. Fortunately, the sub-formula $\forall \alpha : \alpha \in \text{mod}_{proc}[\overline{a}/\overline{x}] \rightarrow \alpha \in \text{mod}_{prog}$ captures this property, ensuring that the outer procedure does not exceed its own mod specification, taking advantage of the interval-union structure of the mod. Since all the modifications (even atomic ones) are done by means of procedure calls,

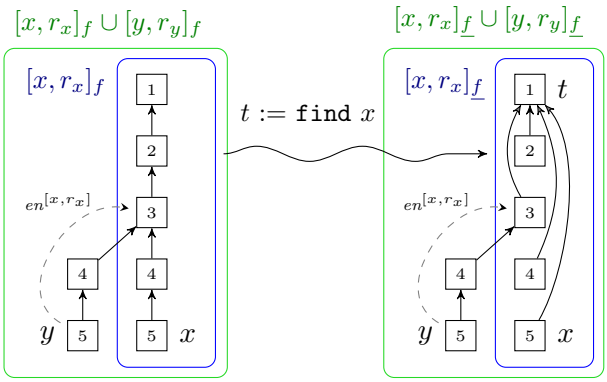

Figure 5.15: An example invocation of `find` inside `union`.

this ensures that no edges incident to nodes outside mod are changed.

Proposition 7. *The rule for wlp⟦⟧ of call statements is sound and complete, that is, when proc is a procedure with specification as in Fig. 5.14, called in the context of prog whose mod-set is* mod*:*

$$\begin{gathered}
\underline{M} \models wlp[\![r := proc(\overline{a})]\!](Q) \\
\Updownarrow \\
\underline{M} \models P_{proc}[\overline{a}/\overline{x}] \wedge \mathrm{mod}^{\underline{M}}_{proc} \subseteq \mathrm{mod}^{\underline{M}} \wedge \\
\forall M : \left(\underline{M}/M \models Q_{proc}[\overline{a}/\overline{x}] \wedge \underline{M} \oplus M \subseteq \mathrm{mod}^{\underline{M}}_{proc}\right) \\
\Rightarrow M[r \mapsto \mathrm{retval}^M] \models Q
\end{gathered} \tag{5.7}$$

5.3.2.3 Reducing Function Symbols

Notice that when we apply the adaptation rule for AE^{AR}, as discussed above, it introduces a new function symbol en^{mod} depending on the concrete mod-set of the called procedure. This introduces a complication: the mod-sets of separate procedure calls may differ from the one of the top procedure, hence multiple applications of Table 5.2 naturally require a separate function symbol en^{mod} for every such invocation. Consider for example the situation of `union` making two invocations to `find`. In Fig. 5.15 one can see that the mod of `union` is $[x, r_x]_f \cup [y, r_y]_f$, while the mod of the first call `t := find(x)` is $[x, r_x]_f$, which may be a proper subset of the former. The mod of the second invocation is $[y, r_y]_f$, which may overlap with $[x, r_x]_f$.

To meet the requirement of AE^{AR} concerning the function symbols, we observe that: (a) the amount of sharing that any particular function call creates, as well as the entire

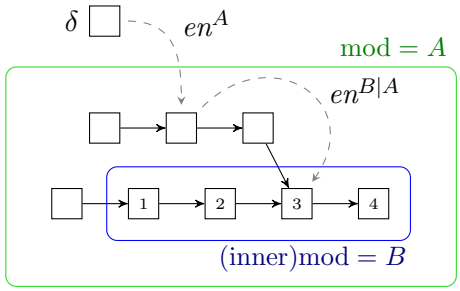

Figure 5.16: The inner en^{mod} is constructed from the outer one by composing with an auxiliary function $en^{B|A}$.

call, is bounded, and we can pre-determine a bound for it; (b) the modification set of any sub-call must be a subset of the top call, as otherwise it violates the obligation not to change any edge outside mod. These two properties allow us to express the functions en^{mod} of the sub-calls using en^{mod} of the top procedure and extra intermediate functions with bounded image. Thus, we replace all of the function symbols en^S introduced by adapt$[S]$ for different S's, with a single (global, idempotent) function symbol together with a set of bounded function symbols.

Consider a statement $r := proc(\overline{a})$ in a procedure $prog$. Let A denote the mod-set of $prog$, and B the mod-set of $proc$. We show how en^B can be expressed using en^A and one more function, where the latter has a bounded range. We define $en^{B|A} : A \setminus B \to B$ a new function that is the restriction of en^B to the domain $A \setminus B$. $en^{B|A}$ is defined as follows:

$$en^B(\sigma) \stackrel{\text{def}}{=} \begin{cases} en^A(\sigma) & en^A(\sigma) \in B \\ en^{B|A}(en^A(\sigma)) & \text{otherwise} \end{cases} \tag{5.8}$$

Using equality the nesting of function symbols can be reduced (without affecting the quantifier alternation).

Consult Fig. 5.16; notice that $en^{B|A}(\sigma)$ is always either:

- The beginning s_i of one of the intervals $[s_i, t_i]_f$ of B (such as $\boxed{1}$ in the figure);
- A node that is **shared** by backbone pointers from two nodes in A (such as $\boxed{3}$);
- The value null.

A bound on the number of s_i's is given in the modular specification of $proc$. A bound on the number of shared nodes is given in the next subsection. This bound is effective for all the procedure calls in $prog$; hence $en^{B|A}$ can be used in the restricted

logic AE^{AR}.

5.3.2.4 Bounding the Amount of Sharing

We show that under the restrictions of the specification given in Section 5.3.1, the number of shared nodes inside mod — that is, nodes in mod that are pointed two by more than one f-pointer of other nodes in mod — has a fixed bound throughout the procedure's execution.

Consider a typical loop-free program *prog* containing calls of the form $v_i := proc_i(\overline{a}_i)$. Assume that the mod-set of *prog* is a union of k intervals. We show how to compute a bound on the number of shared nodes inside the mod-set. Since there are k start points, at most $\binom{k}{2}$ elements can be shared when *prog* starts executing. Each sub-procedure invoked from *prog* may introduce, by our restriction, at most as many shared nodes as there are local variables in the sub-procedure. Therefore, by computing the sum over all invocation statements in *prog*, plus $\binom{k}{2}$, we get a fixed bound on the number of shared nodes inside the mod-set.

$$N_{\text{shared}} = k + \binom{k}{2} + \sum_{proc_i} |Pvar_{proc_i}|$$

$Pvar^{proc_i}$ signifies the set of local variables in the procedure $proc_i$. Notice that if the same procedure is invoked twice, it has to be included twice in the sum as well.

5.3.3 Verification Condition for the Entire Procedure

Since every procedure on its own is loop-free, the verification condition is straightforward:

$$vc[prog] = P_{prog} \rightarrow wlp[\![prog]\!](Q_{prog} \wedge \text{“}shared \subseteq Pvar\text{”})$$

where “$shared \subseteq Pvar$” is a shorthand for the (AE^R) formula:

$$\begin{gathered} \forall \alpha, \beta, \gamma \in \text{mod} : E_f(\alpha, \gamma) \wedge E_f(\beta, \gamma) \rightarrow \\ \alpha = \beta \vee \bigvee_{v \in Pvar} \gamma = v \end{gathered} \tag{5.9}$$

See Eq (3.6) for the definition of E_f. This It expresses the obligation mentioned in Section 5.1.2 that all the shared nodes in mod should be pointed to by local variables, effectively limiting newly introduced sharing to a bounded number of memory locations.

Now $vc[prog]$ is expressed in AE^{AR}, and it is valid if-and-only-if the program meets its specification. Its validity can be checked using effectively-propositional logic according to Section 5.3.1.

5.4 Empirical Results

5.4.1 Implementation Details

A VC generator described in Section 5.3.2 is implemented in Python, and PLY (Python Lex-Yacc) is employed at the front-end to parse modular recursive procedure specifications as defined in Section 5.3.1. The tool checks that the pre and the post-conditions are specified in AF^R and that the modset is defined. SMT-LIB v2 [5] standard notation is used to format the VC and to invoke Z3. As before, the validity of the VC can be checked by providing its negation to Z3. If Z3 exhibits a satisfying assignment then that serves as counterexample for the correctness of the assertions. This means that either the procedure's post-condition is not met, or that some precondition in one of the sub-calls is violated. If no satisfying assignment exists, then the generated VC is valid, and therefore the program satisfies the assertions.

The output model/counterexample (S-Expression), if one is generated, is then also parsed and f^* is evaluated on all pairs of nodes. This structure represents the state of the program either at the input or at the beginning of a loop iteration: running the program from this point will violate one or more invariants. To provide feedback to the user, f is recovered by computing Eq (3.6), and then the `pygraphviz` tool is used to visualize and present to the user a directed graph, whose vertices are nodes in the heap, and whose edges are the f pointer fields.

5.4.2 Verification Examples

We have written modular specifications for the example procedures shown in Table 5.4. We are encouraged by the fact that it was not difficult to express assertions in AF^R for these procedures. The annotated examples and the VC generation tool are publicly available. We picked examples with interesting cutpoints to show the benefits of the modular approach relative to the techniques presented in Chapter 3.

To give some account of the programs' sizes, we observe the program summary specification given as pre- and postcondition, count the number of atomic formulas in

UF: find, UF: union	— Implementation of a Union-Find dynamic data structure.
SLL: filter	— Takes a linked list and deletes all elements not satisfying some predicate C.
SLL: quicksort	— Sorts a linked-list in-place using the Quicksort algorithm.
SLL: insert-sort	— Creates a new, sorted linked-list from a given list by repeatedly running `insert` on the elements of the input list.

Table 5.3: Description of some pointer manipulating programs verified by our tool.

Benchmark	Formula size					Solving time
	P,Q		mod	VC		
	#	∀	#	#	∀	(Z3)
SLL: filter	7	2	1	217	6	0.48s
SLL: quicksort	25	2	1	745	9	1.06s
SLL: insert-sort	21	2	1	284	11	0.37s
UF: find	13	2	1	203	6	0.40s
UF: union	20	2	2	188	6	1.39s

Table 5.4: Implementation Benchmarks; P,Q — program's specification given as pre- and post-condition, mod— mod-set, VC — verification condition, # — number of atomic formulas/intervals, ∀ — quantifier nesting The tests were conducted on a 1.7GHz Intel Core i5 machine with 4GB of RAM, running OS X 10.7.5. The version of Z3 used was 4.2, complied for 64-bit Intel architecture (using `gcc` 4.2, LLVM). The solving time reported is wall clock time of the execution of Z3.

each of them, and note the depth of quantifier nesting; all our samples had only universal quantifiers in the specification. We did the same for the generated VC; naturally, the the VC is much larger than the specification even for small programs. Still, the time required by Z3 to prove that the VC is valid is short.

Thanks to the fact that FOL-based tools, and in particular SAT solvers, permit multiple relation symbols we were able to express ordering properties in sorted lists, and thus in the sorting routines implementing Quicksort and insertion-sort.

5.4.3 Buggy Examples

We also applied the tool to erroneous programs and programs with incorrect assertions. The results, including run-time statistics and formula sizes, are reported in Table 5.5. In addition, we measured the size of the model generated, by observing the size of

Benchmark (+ Nature of defect)	Formula size P,Q #	∀	VC #	∀	Solving time (Z3)	C.e. size (|L|)
UF: find Incorrect handling of corner case	27	3	201	6	1.60s	2
UF: union Incorrect specification	19	2	186	6	0.70s	8
SLL: filter Uncontrolled sharing	36	4	317	6	0.49s	14
SLL: insert-sort Unmet call precondition	21	2	283	9	0.88s	8

Table 5.5: Information about benchmarks that demonstrate detection of several kinds of bugs in pointer programs. In addition to the previous measurements, the last column lists the size of the generated counterexample in terms of the number of vertices — linked-list or tree nodes.

the generated domain—which reflects the number of nodes in the heap. As expected, Z3 was able to produce concrete counterexamples of reasonable size, providing output that is readable for the programmer and useful for debugging. Since these are slight variations of the correct programs, size and running time statistics are similar.

5.5 Related Work for Chapter 5

Modular Verification. The area of modular procedure specification is heavily studied. Many of these works require that the user declare potential changes similar to the modset (e.g., see [4, 41, 66, 71]). The frame rule of separation logic [35] naturally supports modular reasoning where the separating conjunction combines the local post-condition with the assertion at the call site. Unlike separation, reachability is a higher abstraction which relies on type correctness and naturally abstracts operations such as garbage collection. Nevertheless, in Section 6.2 we show that it can also deal with explicit memory reclamations.

We believe that our work pioneers the use of an effectively propositional logic, which is a weak logical fragment, to perform modular reasoning in a sound and complete way.

Our adaptation rule is more complex than the frame rule as it automatically updates reachability.

Cutpoints. Rinetzky et al. [56] introduce cutpoint objects which are objects that can reach the area of the heap accessible by the procedure without passing through objects directly pointed-to by parameters. Cutpoints complicate program reasoning. They are used in model checking [1] and static analysis [24, 57]. Examples such as the ones in [58] which include (unbounded) cutpoints from the stack are handled by our method without any changes. These extra cutpoints cannot change the reachabilty and thus have no effect. Interestingly, we can also handle certain programs which manipulate unbounded cutpoints. Instead, we do limit the amount of new sharing in paths which are necessary for the verification. For example, the find procedure shown in Fig. 5.2 includes unbounded sharing which can be created by the client program. A typical client such as a spanning tree construction algorithm will indeed create unbounded sharing. In the future, we plan to verify such clients by abstracting away the pointers inside the union-find tree.

Chapter 6

Discussion

6.1 On the Expressivity Limitations of AF^R

6.1.1 Inversion yielding a non-AF^R formula

We show here that a naïve attempt to encode the summary of a procedure (e.g. the one in Fig. 3.1) immediately leads to formula that are not AF^R.

Take I'_9 (Eq (3.3)), and substitute occurrences of $\langle n \rangle$ using the inversion formula Eq (3.6). We obtain—

$$\forall \alpha, \beta : \big(\alpha\langle n^+\rangle\beta \wedge \forall\gamma : \alpha\langle n^+\rangle\gamma \rightarrow \beta\langle n^*\rangle\gamma\big) \Leftrightarrow$$
$$\big(\alpha\langle n_0^+\rangle\beta \wedge \forall\gamma : \alpha\langle n_0^+\rangle\gamma \rightarrow \beta\langle n_0^*\rangle\gamma\big)$$

which, when converted to Prenex normal form yields the *non-*AF^R formula

$$\begin{aligned}
\forall\alpha,\beta : \exists\gamma_1\forall\gamma_1' \quad \exists\gamma_2\forall\gamma_2' : \quad & \big(\alpha\langle n^+\rangle\beta \wedge (\alpha\langle n^+\rangle\gamma_1 \rightarrow \beta\langle n^*\rangle\gamma_1) \rightarrow \\
& \quad \alpha\langle n_0^+\rangle\beta \wedge (\alpha\langle n_0^+\rangle\gamma_1' \rightarrow \beta\langle n_0^*\rangle\gamma_1')\big) \\
\wedge \quad & \big(\alpha\langle n_0^+\rangle\beta \wedge (\alpha\langle n_0^+\rangle\gamma_2 \rightarrow \beta\langle n_0^*\rangle\gamma_2) \rightarrow \\
& \quad \alpha\langle n^+\rangle\beta \wedge (\alpha\langle n^+\rangle\gamma_2' \rightarrow \beta\langle n^*\rangle\gamma_2')\big)
\end{aligned}$$

6.1.2 Formulas not expressible in AF^R

Unbounded cutpoints. In Section 3.2 we saw that with the assumption of ownership, the postcondition of *reverse* could be expressed as an AF^R formula. In contrast,

```
void correl_lists(int sz) {
  Node c = null; Node d = null;
  while  (sz > 0)   {
    Node t = new Node();
    t.next = c; c = t;
    t = new Node();
    t.next = d; d = t;
  }
  while (c != null) {
    c = c.next;d = d.next;
  }
}
```

Figure 6.1: A simple Java program that creates two correlated lists.

if we assume an unbounded number of cutpoints then Eq (3.10) must be changed to

$$\forall \alpha, \beta : \alpha\langle n^* \rangle \beta \Leftrightarrow \left\{ \begin{array}{ll} \beta\langle n_0^* \rangle \alpha & h_0\langle n_0^* \rangle \alpha \wedge h_0\langle n_0^* \rangle \beta \\ \alpha\langle n_0^* \rangle \beta & \neg h_0\langle n_0^* \rangle \alpha \wedge \neg h_0\langle n_0^* \rangle \beta \\ \textbf{false} & h_0\langle n_0^* \rangle \alpha \wedge \neg h_0\langle n_0^* \rangle \beta \\ \exists \gamma : \alpha\langle n_0^* \rangle \gamma \wedge & \\ \quad \neg h_0\langle n_0^* \rangle \gamma \wedge \beta\langle n_0^* \rangle n_0(\gamma) & \neg h_0\langle n_0^* \rangle \alpha \wedge h_0\langle n_0^* \rangle \beta \end{array} \right\} \tag{6.1}$$

The first three cases are the same as in Eq (3.10). The last case considers the situation where α is outside the list while β is within the list. For α to reach β in the postcondition, it must be the case that there exists a node γ such that γ is outside the list but its successor is within the list and reachable from β. The formula, however, introduces alternation of $\exists$ inside $\forall$ and the use of the function symbol n so it is outside AE^R (and thus outside AF^R).

Correlations Between Data Structures. Another example of a non-AF^R formula mentioned concerns programs that manipulate two lists of the same length. The program in Fig. 6.1 demonstrates a case where a weak logic is not enough to prove the absence of null dereference in a pointer program. The first while loop creates two lists of length sz. Then, taking advantage of the equal lengths, it traverses the first list — the one pointed to by c — while at the same time advancing the pointer d.

Since each iterator advances one step, the second loop preserves an invariant that the lists at c and d are of the same length. Hence, as long as c is not *null*, it guarantees that d is not *null* either. Unfortunately, such an invariant requires an inductive definition which is well outside of AF^R:

$$\begin{aligned} eqlen(x,y) \quad &\stackrel{\text{def}}{=} \quad (x = null \wedge y = null) \quad \vee \\ &\qquad \big(x \neq null \wedge y \neq null \wedge eqlen(n(x), n(y))\big) \end{aligned}$$

6.2 Extensions

Doubly-linked List and Nested Lists. To verify a program that manipulates a doubly-linked list, all that needs to be done is to duplicate the analysis we did for n, for a second pointer field *prev*. As long as the only atomic formulas used in assertions are $\alpha\langle n^*\rangle\beta$ and $\alpha\langle prev^*\rangle\beta$ (and not, for example, $\alpha\langle(n|prev)^*\rangle\beta$), providing the substitutions for atomic formulas in Table 3.3 would not get us outside of the class AE^R. In particular, we have verified the doubly-linked list property:

$$\forall\alpha,\beta : h\langle n^*\rangle\alpha \wedge h\langle n^*\rangle\beta \implies (\alpha\langle n^*\rangle\beta \Leftrightarrow \beta\langle prev^*\rangle\alpha).$$

In fact we can verify nested lists and, in general, lists with arbitrary number of pointer fields as long as reachability constraints are expressed using only one function symbol at a time, like in the case of *next* and *prev* above.

Cycles. For data structures with a single pointer, the acyclicity restriction may be lifted by using an alternative formulation that keeps and maintains more auxiliary information [28, 40]. Instead of keeping track of just n^*, we instrument the edge addition operation with a check: if the added edge is about to close a cycle, then instead of adding the edge, we keep it in a separate set M of "cycle-inducing" edges. Two properties of lists now come into play: (1) The number of cycles reachable from program variables, and hence the size of M, is bounded by the number of program variables; (2) Any path (simple or otherwise) in the heap may utilize at most one of those edges, because once a path enters a cycle, there is no way out. In all assertions, therefore, we replace $\alpha\langle n^*\rangle\beta$ with: $\alpha\langle n^*\rangle\beta \vee \bigvee_{\langle u,v\rangle\in M}(\alpha\langle n^*\rangle u \wedge v\langle n^*\rangle\beta)$. Notice that it is possible to construct

All lists are acyclic	$sll(n^*) \wedge sll(m^*)$
No sharing between lists	$\forall \alpha, \beta, \gamma : h\langle n^* \rangle \alpha \wedge \alpha \langle m^* \rangle \beta \wedge h\langle n^* \rangle \gamma \wedge \gamma \langle m^* \rangle \beta \implies \alpha = \gamma$
Hierarchy	$\forall \alpha, \beta, \gamma : \alpha \neq \beta \wedge \beta \neq \gamma \wedge \alpha \langle m^* \rangle \beta \implies \neg \beta \langle n^* \rangle \gamma$
Cycle edge	$\forall \alpha, \beta, \gamma : \alpha \neq \beta \wedge \alpha \neq \gamma \wedge \alpha \langle n^* \rangle \beta \implies \neg \alpha \langle c \rangle \gamma$ $\forall \alpha, \beta : \beta \langle c \rangle \alpha \implies h\langle n^* \rangle \alpha \wedge \alpha \langle m^* \rangle \beta$

Table 6.1: Properties of a list of cyclic lists expressed in AF^R

this formula thanks to the bound on the size of M; otherwise, an existential quantifier would have been required in place of the disjunction.

Cycles can also be combined with nesting, in such a way as to introduce an unbounded number of cycles. To illustrate this, consider the example of a linked list beginning at h and formed by a pointer field which we shall denote n, where each element serves as the head of a singly-linked cycle along a second pointer field m. This is basically the same as in the case of acyclic nested lists, only that the last node in every sub-chain (a list segment formed by m) is connected to the first node of that same chain.

One way to model this in a simple way is to assume that the programmer designates the last edge of each cycle; that is, the edge that goes back from the last list node to the first. We denote this designation by introducing a ghost field named c. This cycle-inducing edge is thus labeled c instead of m.

Properties of the nested data structure can be expressed with AF^R formulas as shown in Table 6.1. "Hierarchy" means that the primary list is contiguous, that is, there cannot be n-pointers originating from the middle of sub-lists. "Cycle edge" describes the closing of the cyclic list by a special edge c.

We were able to verify the absence of memory errors and the correct functioning of the program `flatten`, shown in Fig. 6.2.

Bounded Sharing. Arbitrary sharing in data structures is hard, because even in lists, any node of the list may be shared (that is, have more than one incoming edge). In this case we have to use quantification since we do not know in advance which node in the list is going to be a cutpoint for which other nodes. However, when the *entire* heap consists solely of lists, the quantifier may be replaced with a disjunction if we take into account that there is a bounded number of program variables, which can serve

```
Node flatten(Node h) {
  Node i = h, j = null;
  while (i != null) I1 {
    Node k = i;
    while (k != null) I2 {
      j = k; k = k.m;
    }
    j.c = null;
    i = i.n; j.m = null; j.m = i;
  }
  j.c = null; j.c := h;
  return h;
}
```

Figure 6.2: A program that flattens a hierarchical structure of lists into a single cyclic list.

Command	Pre	Mod	Post
`ret = new()`	$free(s)$	$\emptyset$	$ret = s \wedge \neg free(s)$
`access y`	$\neg free(y)$	$\emptyset$	
`free(y)`	$y \neq \text{null} \wedge \neg free(y)$	$\emptyset$	$free(y)$

Table 6.2: The specifications of atomic commands for resource allocations in a C-like language.

as the heads of lists, and any two lists have at most one cutpoint. Such heaps when viewed as graphs are much simpler than general DAGs, since one can define in advance a set of *constant symbols* to hold the edges that induce the sharing; for example, if we have one list through the nodes $x \to u_1 \to u_2$ and a second list through $y \to v_1 \to v_2$, all distinct locations, then adding an edge $u_2 \to v_1$ would create sharing, as the nodes v_1, v_2 become accessible from both x and y. This technique is also covered by Hesse [28].

Explicit Memory Management Table 6.2 updates the specification of atomic commands (provided in Table 5.1) to handle explicit memory management operations. For simplicity, we follow ANSI C semantics but do not handle arrays and pointer arithmetic. The allocation statement assigns a freed location denoted by s to ret and sets its value to be non-freed. All accesses to memory locations pointed-to by y in statements `ret = y.f`, `y.f = x`, and `x = y` are required to access non-freed memory. Finally, `free(y)` sets the free predicate to true for the node pointed-to by y. As a result, all the nodes reachable from y cannot be accessed.

Notice that it is possible to enforce interesting safety properties naturally using

AF^R formulas such as the absence of memory leaks. For example a precondition for `free(x)` can be

$$\forall \alpha : y \langle f^* \rangle \alpha \wedge \alpha \neq y \wedge \alpha \neq \text{null} \rightarrow \mathit{free}(y)$$

Chapter 7

Conclusion

In this thesis we investigated complex properties of programs, involving high-order concepts as transitive closure of pointer links and its derivatives such as sharing and entry-point. It is surprising that such properties can be expressed and handled by a logic based on EPR, which is among the weakest fragments of first-order logic. A key insight to enable this was the idea of "inversion": instead of a function symbol for *next* and a derived binary relation $next^*$ for its transitive closure, formalize the properties of $next^*$ and have *next* derived from it instead. As previously noted, this is analogous to reasoning in the natural numbers using a successor function *succ* (which requires constant use of induction) vs. reasoning with $\leq$.

The results shed some light on the complexity of reasoning about programs that manipulate linked data structures such as singly- and doubly-linked lists. The invariants in many of these programs can be expressed without quantifier alternation. Alternations are introduced by unbounded cutpoints and reasoning about more complicated directed acyclic graphs. Furthermore, for programs manipulating general graphs higher order reasoning may be required.

Compared to past work on shape analysis, our approach (i) is based on full predicate abstraction, (ii) makes use of standard theorem proving techniques, (iii) is capable of reporting concrete counterexamples, and (iv) is based on property-directed reachability. The experimental evaluation in Section 4.3 illustrates these four advantages of our approach. The algorithm is able to establish memory-safety and preservation of data-structure invariants for all of the examples, using only a handful of simple predicates. This result may look somewhat surprising because earlier work on shape analysis that employed the same predicates [26] failed to prove these properties. One reason is

that [26] only uses positive and negative combinations of these predicates, whereas our algorithm uses arbitrary Boolean combinations of predicates.

A crucial method for simplifying the reasoning about linked data structures is partitioning programs into smaller pieces, where each piece manipulates part of the heap. The part that a sub-program manipulate usually has a bounded number of entry and exit points. This thesis slightly generalizes by reasoning about a potentially unbounded number of entry points, as demonstrated by `find` and `union`. Only the *new* sharing that is introduced by the procedure should be bounded. Notice that this unboundedness supports modularity: even in the case where in every particular call context there is a bounded number of paths (e.g. when there is a bounded number of roots in the heap), the bound is not known in advance, therefore the programmer has to prepare for an unbounded number of cases.

It is important to note that the adaptation rule of Section 5.2 adds expressive power to verifying programs: it is in general impossible for the programmer to define, in AF^R, a modular specification for all the procedures. Generation of a verification condition requires coordination between the separate call sites as mentioned above, in particular taking note of potential sharing. This coordination requires per-call-site instantiation, which, thanks to having the adaptation rule in the framework, is done automatically.

Finally we remark that there is a trade-off between the complexity of the mod-set and of the post-condition: defining a simpler, but larger mod may cause the post-condition to become more complicated, sometimes not even AF^R-expressible. Also notice that if mod $= V$ (the entire heap), modular reasoning becomes trivial since it can be done by relational composition, but this puts the burden of writing the most complete post-conditions on the programmer, which we suspect is not always possible in our limited logic.

Therefore, we believe that this thesis is an important step towards the ability to handle real-life programs in a scalable way.

There are some issues still open. As we have seen, there is an interesting class of programs that work on linked data structures with pointers and admit EPR inductive loop invariants. Since EPR is a countable set of formulas, and from the decision procedure developed in the previous chapters, the problem of checking whether a program has such a loop invariant becomes r.e. by a simple generate-and-test approach (of course, in general, the problem of checking a single loop invariant is undecidable,

and the problem of finding a loop invariant is not r.e.). Is there a better way to answer this question? The method represented in Chapter 4 for invariant inference allows limited use of quantifiers, restricted by the parameter templates. It would be useful to learn new quantified predicates and refine the abstractions (e.g. using CEGAR); this would require less from the programmer and would allow more instances where fully-automated reasoning on loops can be applied. On a broader scale, we raise a theoretical question of whether an EPR invariant can be discovered automatically, and whether the existence of an EPR invariant is decidable. We conjecture, however, that the answer would be negative, although a proof is yet to be discovered.

The solution might be a heuristic for EPR inductive loop invariant inference that would be sound but not complete; while not providing a conclusive solution, a heuristic has a high chance of working well in practice because naturally-occurring programs are usually "well-behaved" and have a simple argument that proves their correctness. Moreover, it can be argued that having a simple loop invariant is a good programming trait by its own right and that programmers should make an effort to write such programs. In any case, it is obvious that the programs presented in undecidability proofs are highly unnatural and there is no practical need to handle such instances.

To design such heuristics, a larger base of benchmarks should be obtained and inspected to characterize what common programming idioms exist, then exploit properties of these idioms to handle variants of the benchmarks in different contexts.

On the aspects of modular programming and reasoning, we still do not know the bounds of modular reasoning and in which situations it can be used. The restriction of bounded sharing (Section 5.3.2.4) may be too severe for some cases, and can probably be avoided. Also, it is interesting to investigate the differences and similarities of handling procedure calls within loops vs. handling recursion as is done in this thesis.

Finally, the success we had with reasoning of transitive closure gives hope that this kind of reasoning can also be applied for the field of synthesis, where invariants are given and the code is automatically generated, rather than being checked against the specifications. This can be a win-win situation, since sometimes generation of code may even be easier than verification, and the resulting code is correct by construction.

Appendix A

Logical Proofs

Definition 15 (Vocabulary)**.** *A* ***vocabulary*** $\mathcal{V}$ *is a triple* $\langle \mathcal{C}, \mathcal{F}, \mathcal{R} \rangle$ *where* $\mathcal{C}$ *is a finite set of constant symbols,* $\mathcal{F}$ *is a finite set of function symbols where each* $f \in \mathcal{F}$ *has a fixed arity* $ar(f)$*, and* $\mathcal{R}$ *is a finite set of relation symbols where each* $r \in \mathcal{R}$ *has a fixed arity* $ar(r)$*.*

A.1 Reductions between Logics

We now state the reduction from the problem of checking validity of an AE^R formula to effectively-propositional logic for a specific function n. It can be generalized for any fixed number of function symbols in order to handle, for example, doubly-linked lists or nested lists. First we show that for *acyclic functions over finite domains*, there is a one-to-one correspondence between n and n^*, as hinted previously by Eq (3.6).

Let L be the set of memory locations ("objects"). Let $\alpha \in L$ be some node, then the set of nodes reachable from α, that is, $W = \{\beta \mid \alpha\langle n^* \rangle \beta\}$, is linearly ordered via n^*. Thus α has a unique successor — namely, the minimal node of $W \setminus \{\alpha\}$. This gives rise to the following lemma:

Lemma 3. *For* $n : L \to L \cup \{null\}$*, which is acyclic,*

$$\beta = n(\alpha) \iff \alpha\langle n^+ \rangle \beta \wedge \forall \gamma : \alpha\langle n^+ \rangle \gamma \implies \beta\langle n^* \rangle \gamma$$

for any $\alpha, \beta \in L$*.*

Proof. First direction (⇒): Let $\beta = n(\alpha)$. Trivially $\alpha\langle n^+ \rangle \beta$. Assume some $\gamma \in L$ such that $\alpha\langle n^+ \rangle \gamma$, then $n(\alpha)\langle n^* \rangle \gamma$; Hence $\beta\langle n^* \rangle \gamma$.

Second direction ($\Leftarrow$): Let $\alpha\langle n^+\rangle\beta$ and $\alpha\langle n^+\rangle\gamma \implies \beta\langle n^*\rangle\gamma$ for any $\gamma \in L$. From the first clause, $n(\alpha)\langle n^*\rangle\beta$. From the second, since $\alpha\langle n^+\rangle n(\alpha)$, follows $\beta\langle n^*\rangle n(\alpha)$. Due to acyclicity, $\beta = n(\alpha)$. □ □

Relying on the fact that L is finite, the right-hand side of the lemma necessarily defines a total function (which is, in fact, computable). We can use this fact to simulate reachability constraints in first-order logic. For this purpose, we use the formula φ' from Eq (3.7) (Section 3.2).

Proposition 2 (Simulation of AE^R) *Consider an AE^R formula φ over vocabulary $\mathcal{V} = \langle \mathcal{C}, \{n\}, \mathcal{R}\rangle$. Let $\varphi' \stackrel{\text{def}}{=} \varphi[\widehat{n^*}(t_1, t_2)/t_1\langle n^*\rangle t_2]$. Then φ' is a FO formula over vocabulary $\mathcal{V}' = \langle \mathcal{C}, \emptyset, \mathcal{R} \cup \{\widehat{n^*}\}\rangle$ and φ is simulated by $\Gamma_{linOrd} \to \varphi'$ where Γ_{linOrd} is the formula in Table 3.2.*

Proof. We need to show that φ is valid $\iff$ φ' is valid.

First direction ($\Rightarrow$): Suppose that φ is true on all appropriate structures. Let $A' \in \text{STRUC}(\mathcal{V}')$ be an arbitrary *finite* structure for φ', with domain L such that $A' \models \Gamma_{\text{linOrd}}$. Define n^A as in Lemma 3 — this is well-defined since L is finite. We got a structure A for φ, so $A \models \varphi$, and $(n^A)^* = \widehat{n^*}^{A'}$. Therefore, $A' \models \varphi'$.

Second direction ($\Leftarrow$): Conversely, suppose that $\Gamma_{\text{linOrd}} \implies \varphi'$ is true on all appropriate structures, and let $A \in \text{STRUC}(\mathcal{V})$ be an arbitrary structure for φ; By setting $\widehat{n^*}^{A'} = (n^A)^*$ we get A', which is a model of Γ_{linOrd}; hence from the assumption, $A' \models \varphi'$; therefore $A \models \varphi$. □ □

A.2 Program Semantics

This section of the appendix refers to the definitions of Table 2.2, Table 2.3 (Section 2.2) and Table 3.3 (Section 3.4).

Proposition 3 (VCs in AE^R) *For every program S whose precondition P, postcondition Q, branch conditions, loop conditions, and loop invariants are all expressed as AF^R formulas, $VC_{gen}(\{P\}S\{Q\}) \in AE^R$.*

Proof. Follows from closure properties of AE^R, and from the fact that AE^R is closed under *wlp*, The proofs of which immediately follow. In particular, in $VC_{aux}(\mathsf{while}\ldots)$, the subformulas $[\![B]\!]$ and I are AF^R and Q is AE^R. Thus $I \wedge [\![B]\!] \to Q$ is AE^R. Similarly for the $\neg[\![B]\!]$ case. □

The following proposition summarizes the properties of formulas which are used to guarantee that we can generate AF^R formulas for arbitrary procedures manipulating singly and doubly linked lists with AF^R specified invariants.

Proposition 8 (Closure of AE^R Formulas)**.** *Let af be a closed AF^R formula, qf be a QF^R formula. Let φ_1 and φ_2 be closed AE^R formulas. Let a be an atomic subformula in φ_1 and let $\varphi_1[qf/a]$ denote the substitution of a in φ_1 by qf. Let c be a constant. Then, the following formulas are all AE^R formulas:* ***disjunction****: $\varphi_1 \vee \varphi_2$;* ***conjunction****: $\varphi_1 \wedge \varphi_2$;* ***$AF^R$-implication****: $af \implies \varphi_1$;* ***QF^R-substitution****: $\varphi_1[qf/a]$;* ***generalization****: $\forall\alpha : \varphi_1[\alpha/c]$.*

Theorem 5. (Soundness and Completeness of the *wlp* rules for heap access—atomic statements) *For every state σ be a structure with the vocabulary $\langle Pvar, \{n\}, \mathcal{R}\rangle$ over a set of locations L, for every atomic heap access atomic statement S, $\sigma \models wlp[\![S]\!](Q)$ if and only if $[\![S]\!]\sigma \models Q \wedge ac$.*

Proof. We will prove the theorem in terms of the operational semantics of the atomic statements.

Let $\sigma, \sigma' \in \Sigma$ be program states such that $\sigma \models P \wedge ac$.

So n^σ is a function without cycles.

Consider the two atomic commands whose semantics update n — this is based almost completely on [28]:

$S = x.n := y$: Assume that $n^\sigma(x^\sigma) = null$. Then $n^{[\![S]\!]\sigma}(x^\sigma) = y^\sigma$, and for any $\alpha \neq x^\sigma$, $n^{[\![S]\!]\sigma}(\alpha) = n^\sigma(\alpha)$. In this case, $\alpha\langle n^{[\![S]\!]\sigma *}\rangle\beta \iff \alpha\langle n^{\sigma *}\rangle\beta \vee (\alpha\langle n^{\sigma *}\rangle x^\sigma \wedge y^\sigma\langle n^{\sigma *}\rangle\beta)$, so via our assumption,

$$[\![S]\!]\sigma \models Q \iff \sigma \models Q[\alpha\langle n^*\rangle\beta \vee (\alpha\langle n^*\rangle x \wedge y\langle n^*\rangle\beta)/\alpha\langle n^*\rangle\beta)]$$

Because σ and $[\![S]\!]\sigma$ differ only in n (values of all program variables are the same), that is $[\![S]\!]\sigma = \sigma[n \mapsto n^{[\![S]\!]\sigma}]$.

$S = x.n := null$: Similarly, we have $n^{[\![S]\!]\sigma}(x^\sigma) = null$ and for any $\alpha \neq x^\sigma$, $n^{[\![S]\!]\sigma}(\alpha) = n^\sigma(\alpha)$. Therefore in this case $\alpha\langle n^{[\![S]\!]\sigma *}\rangle\beta \iff \alpha\langle n^{\sigma *}\rangle\beta \wedge (\neg\alpha\langle n^{\sigma *}\rangle x^\sigma \vee \beta\langle n^{\sigma *}\rangle x^\sigma)$ (here it is important that we know that n is without cycles). Hence,

$$[\![S]\!]\sigma \models Q \iff \sigma \models Q[\alpha\langle n^*\rangle\beta \wedge (\neg\alpha\langle n^*\rangle x \vee \beta\langle n^*\rangle x)/\alpha\langle n^*\rangle\beta]$$

Consider the atomic command whose semantics read the value of n. These do not change n, so safely $\sigma \models ac \iff [\![S]\!]\sigma \models ac$. It is left to check the condition for Q:

$S = x := y.n$: Assume $\ell \in L$ such that $\sigma, A \models P_n[y/s, \alpha/t]$ where $A = [\alpha \mapsto \ell]$ is a variable assignment for a fresh variable a. According to Lemma 3, this means that $n^\sigma(y^\sigma) = \ell$. We then get $[\![S]\!]\sigma = \sigma[x \mapsto n^\sigma(y^\sigma)] = \sigma[x \mapsto \ell]$.

$$[\![S]\!]\sigma \models Q \iff \sigma[x \mapsto \ell] \models Q \iff \sigma, A \models Q[\alpha/x]$$

And since this holds for any A as above,

$$[\![S]\!]\sigma \models Q \iff \sigma \models \forall \alpha : P_n[y/s, \alpha/t] \implies Q[\alpha/x]$$

Finally, consider the atomic command that allocates memory.

$S = x :=$ new: We model the memory allocation as a function $M : \Sigma \to L$ that given a structure σ, returns a non-*null* location $\ell \in L$ such that ℓ is not "used" — in the sense that it cannot be reached from any of the program variables, that is, constant symbols of the set *Pvar*. In this case $[\![S]\!]\sigma = \sigma[x \mapsto M(\sigma)]$. Also,

$$\sigma, \big[\alpha \mapsto M(\sigma)\big] \models \bigwedge_{p \in Pvar \cup \{null\}} \neg p\langle n^* \rangle \alpha$$

Assume $\sigma \models \forall \alpha : \Big(\bigwedge_{p \in Pvar \cup \{null\}} \neg p\langle n^* \rangle \alpha\Big) \implies Q[\alpha/x]$. Hence from the use of $\forall$, we know that $\sigma, \big[\alpha \mapsto M(\sigma)\big] \models \Big(\bigwedge_{p \in Pvar \cup \{null\}} \neg p\langle n^* \rangle \alpha\Big) \implies Q[\alpha/x]$. Combined with M specifications, $\sigma, \big[\alpha \mapsto M(\sigma)\big] \models Q[\alpha/x]$. Hence $[\![S]\!]\sigma \models Q$.

Now assume that $[\![S]\!]\sigma \models Q$. It means it should hold for *any* implementation of M. Let $\ell = M(\sigma)$, then we know that ℓ can be any location such that

$$\sigma, A \models \bigwedge_{p \in Pvar \cup \{null\}} \neg p\langle n^* \rangle \alpha$$

Where $A = \big[\alpha \mapsto \ell\big]$. Also, $\sigma, A \models Q[\alpha/x]$ from the same reasons as before. Since this holds for any such A satisfying the antecendant, we conclude that $\sigma \models \forall \alpha : \Big(\bigwedge_{p \in Pvar \cup \{null\}} \neg p\langle n^* \rangle \alpha\Big) \implies Q[\alpha/x]$.

□

□

A.3 Relative Completeness of IC3 with Predicate Abstraction

Theorem 3 *Given (i) the set of abstraction predicates $\mathcal{P} = \{p_i \in \mathcal{L}\}, 1 \leq i \leq n$ where $\mathcal{L}$ is a decidable logic, and the full predicate abstraction domain $\mathcal{A}$ over $\mathcal{P}$, (ii) the initial condition $Init \in \mathcal{L}$, (iii) a transition relation ρ expressed as a two-vocabulary formula in $\mathcal{L}$, and (iv) a formula $Bad \in \mathcal{L}$ specifying the set of bad states, $PDR_{\mathcal{A}}(Init, \rho, Bad)$ terminates, and reports either*

1. valid *if there exists $A \in \mathcal{A}$ s.t. (i) $Init \rightarrow A$, (ii) A is inductive, and (iii) $A \implies \neg Bad$,*
2. *a concrete counterexample trace, which reaches a state satisfying Bad, or*
3. *an abstract trace, if the inductive invariant required to prove the property cannot be expressed as an element of $\mathcal{A}$.*

Proof. The first two cases are trivial: if $\text{PDR}_{\mathcal{A}}$ terminates returning some $R[j]$, $j < N$, then $Init \rightarrow R[j]$ by virtue of $Init \rightarrow R[0]$ and $R[i] \rightarrow R[i+1]$ for every $i < N$, and $R[j] \rightarrow \neg Bad$ or the check at line line 1 would have failed. Also, $R[j-1] \equiv R[j]$ so $R[j]$ is inductive.

If $\text{PDR}_{\mathcal{A}}$ returns a set of concrete states, then they have to be a concrete counterexample trace, as they are a model of $Init \wedge \rho^{N-j} \wedge (Bad)'^{\times(N-j)}$ (line 2 of $\text{reduce}_{\mathcal{A}}$).

For the third case, we show that if the check on the first line of "reduce" is "sat", then there exists a chain of concrete states, $\sigma_j \quad \sigma_{j+1} \quad \cdots \quad \sigma_N$, such that $\sigma_j \models Init$, $\sigma_N \models Bad$, and for any $j \leq i < N$ there exist two concrete states σ, σ' satisfying:

- $\sigma \in \gamma(\beta_{\mathcal{A}}(\sigma_i))$
- $\sigma' \in \gamma(\beta_{\mathcal{A}}(\sigma_{i+1}))$
- $\langle \sigma, \sigma' \rangle \models \rho$

The key point is that, because the given abstraction can never distinguish any two states in $\gamma(\beta_{\mathcal{A}}(\sigma_i))$, the chain $\sigma_j \quad \sigma_{j+1} \cdots \sigma_N$ cannot be excluded by the abstract domain $\mathcal{A}$, no matter what Boolean combination of the predicates of $\mathcal{P}$ is used. Moreover, the chain $\beta_{\mathcal{A}}(\sigma_j) \quad \beta_{\mathcal{A}}(\sigma_{j+1}) \quad \cdots \quad \beta_{\mathcal{A}}(\sigma_N)$ is an abstract trace that leads from an initial state to an error state.

Notice that the chain above may not be a concrete trace, there can be "breaks" between adjacent σ_is, within the same abstract element.

Construction of $(\sigma_i)_{i=j\ldots N}$: Follow the chain of recursive calls to "reduce" with index values N down to j. The parameter A is always a cube of the form $\beta_{\mathcal{A}}(\sigma)$; take one $\sigma \models A$ for each call, forming a series that we denote by σ_j, σ_{j+1}, etc. We show that this series satisfies the above properties: At each call except the innermost, "reduce" made a recursive call at line 7, which means that $R[j-1] \wedge \rho \wedge (A)'$ was satisfiable; the returned cube A_2 becomes $\beta_{\mathcal{A}}(\sigma_{j-1})$. Let $\langle \sigma, \sigma' \rangle \models R[j-1] \wedge \rho \wedge (A)'$, then $\sigma \models A_2 = \beta_{\mathcal{A}}(\sigma_{j-1})$; $\sigma' \models A = \beta_{\mathcal{A}}(\sigma_j)$; and $\langle \sigma, \sigma' \rangle \models \rho$ as required. □

A.4 Simulation of an Idempotent Function in EPR

We prove Theorem 4, by providing a concrete translation in three steps. First, we translate φ to a $\forall^*\exists^*$-formula φ' as described in Proposition 2, keeping the function symbols $g, h_1, \ldots, h_n$ as is. Similarly to the proof of Proposition 2, one obtains that (a) if φ is valid in all finite structures then φ' is valid in all finite structures; and (b) if φ' is valid then φ is valid.

Second, to eliminate the function symbols $h_1, \ldots, h_n$ we repetitively use the following proposition:

Proposition 9. *Let $N > 0$. Let h be some unary function symbol, and Σ_h be a first-order signature (with equality) that includes the function symbol h. Let Σ_R be the signature obtained from Σ_h by replacing h with a binary relation symbol R_h, and adding N fresh constant symbols $c_1, \ldots, c_N$. Consider an $\exists^*\forall^*$ sentence over Σ_g: $\Phi = \exists x_1, \ldots, x_n. \forall y_1, \ldots, y_m. \Psi$ (where Ψ is quantifier free, over the variables $x_1, \ldots, x_n, y_1, \ldots, y_m$). Suppose that the only terms occurring in Φ that involve h have the form $h(z)$, where z is a variable. Define $T(\Phi)$ to be the following sentence over Σ_R:*

$$\begin{aligned} &\Gamma \wedge \exists x_1, x_1^h, \ldots, x_n, x_n^h. \forall y_1, y_1^h, \ldots, y_m, y_m^h. \\ &\textstyle\bigwedge_{1 \le i \le n} R_h(x_i, x_i^h) \wedge (\bigwedge_{1 \le i \le m} R_h(y_i, y_i^h) \rightarrow \Psi') \end{aligned} \tag{A.1}$$

where:

- Γ *is the conjunction of the following formulas:*
  - $\forall x, y, z. R_h(x, y) \wedge R_h(x, z) \rightarrow y = z$

- $\forall x. \vee_{1 \leq i \leq N} R_h(x, c_i)$

- $x_1^h, \ldots, x_n^h, y_1^h, \ldots, y_m^h$ *are fresh variables.*

- Ψ' *is obtained from* Ψ *by replacing each term of the form* $h(a_i)$ *by* a_i^h *(for* $a \in \{x, y\}$*).*

Then, Φ *is satisfied by a (finite) normal structure interpreting* h *as a function whose image cardinality is at most* N *iff* $T(\Phi)$ *is satisfied by some (finite) normal structures.*

Proof. Suppose that a (finite) normal structure M_h interpreting h as a function whose image cardinality is at most N is a model of Φ. We can obtain a (finite) normal model of $T(\Phi)$ by: (a) interpreting $c_1, \ldots, c_N$ as the elements of the image of h^{M_h}; (b) use the (graph of the) interpretation of h as an interpretation of R_h.

For the converse, suppose that M_R is a (finite) normal model of $T(\Phi)$. We can obtain a (finite) normal model M_h of $T(\Phi)$ by choosing $h^{M_h} = \lambda d \in |M_R|.\iota d' \in |M_R|.\langle d, d' \rangle \in R_h^{M_R}$. This is well-defined since M_R is a model of Γ. It is straightforward to verify that M_h is a model of Φ. □

Finally, to eliminate the function symbol g, we use the following proposition:

Proposition 10. *Let* Σ_g *be a first-order signature (including equality) with only one unary function symbol* g*, and* Σ_R *be the signature obtained from* Σ_g *by replacing* g *with a binary relation symbol* R_g*. Consider an* $\exists^*\forall^*$ *sentence over* Σ_g*:* $\Phi = \exists x_1, \ldots, x_n. \forall y_1, \ldots, y_m. \Psi$ *(where* Ψ *is quantifier free, over the variables* $x_1, \ldots, x_n, y_1, \ldots, y_m$*). Suppose that the only terms occurring in* Φ *are variables and terms of the form* $g(z)$ *(where* z *is a variable). Define* $T(\Phi)$ *to be the following sentence over* Σ_R*:*

$$\begin{aligned} &\Gamma \wedge \exists x_1, x_1^g, \ldots, x_n, x_n^g. \forall y_1, y_1^g, \ldots, y_m, y_m^g. \\ &\textstyle\bigwedge_{1 \leq i \leq n} R_g(x_i, x_i^g) \wedge (\bigwedge_{1 \leq i \leq m} R_g(y_i, y_i^g) \rightarrow \Psi') \end{aligned} \tag{A.2}$$

where:

- Γ_{idem} *is the conjunction of the following formulas:*

  - $\exists x, y. R_g(x, y)$

  - $\forall x, y, z. R_g(x, y) \wedge R_g(x, z) \rightarrow y = z$

– $\forall x, y.R_g(x, y) \to R_g(y, y)$

- $x_1^g, \ldots, x_n^g, y_1^g, \ldots, y_m^g$ *are fresh variables.*
- Ψ' *is obtained from* Ψ *by replacing each term of the form* $g(a_i)$ *by* a_i^g *(for* $a \in \{x, y\}$*).*

Then, Φ *is satisfied by a (finite) normal structure interpreting* g *as an idempotent function iff* $T(\Phi)$ *is satisfied by some (finite) normal structure.*

Proof. The left-to-right direction is clear. Indeed, if a (finite) normal structure M_g interpreting g as an idempotent function is a model of Φ, then we can use the (graph of the) interpretation of g as an interpretation of R_g and obtain a (finite) normal model of $T(\Phi)$. We prove the converse. Suppose that M_R is a (finite) normal model of $T(\Phi)$. Construct a (finite) normal structure M_g (for the signature Σ_g) as follows:

1. $|M_g| = \{d \in |M_R| \ : \ \exists d' \in |M_R|.\langle d, d'\rangle \in R_g^{M_R}\}$
2. $g^{M_g} = \lambda d \in |M_g|.\iota d' \in |M_g|.\langle d, d'\rangle \in R_g^{M_R}$
3. For every k-ary predicate symbol p of Σ_g, $p^{M_g} = p^{M_R} \cap |M_g|^k$ (i.e., p^{M_g} is the restriction of p^{M_R} to the new domain).

$|M_g|$ is non-empty since M_R is a model of $\exists x, y.R_g(x, y)$. To see that g^{M_g} is well-defined, note that if $\langle d, d'\rangle \in R_g^{M_R}$ then both d and d' are in $|M_g|$. Indeed, in this case $d \in |M_g|$ by definition, and since M_R is a model of $\forall x, y.R_g(x, y) \to R_g(y, y)$, we must have $\langle d', d'\rangle \in R_g^{M_R}$, and thus $d' \in |M_g|$ as well. Hence for every $d \in |M_g|$ there exists an element $d' \in |M_g|$ such that $\langle d, d'\rangle \in R_g^{M_R}$. Its uniqueness is guaranteed since M_R is a model of $\forall x, y, z.R_g(x, y) \wedge R_g(x, z) \to y = z$. The fact that g^{M_g} is idempotent directly follows from the fact that $M_R \models \forall x, y.R_g(x, y) \to R_g(y, y)$.

Now, we show that M_g is a model of Φ. Let σ be an $|M_R|$-assignment (assigning elements of $|M_R|$ to the variables), such that $M_R, \sigma' \models \bigwedge_{1 \le i \le n} R_g(x_i, x_i^g) \wedge (\bigwedge_{1 \le i \le m} R_g(y_i, y_i^g) \to \Psi')$ for every $\{y_1, y_1^g, \ldots, y_m, y_m^g\}$-variant σ' of σ. Then, for every $1 \le i \le n$, $\sigma[x_i]$ and $\sigma[x_i^g]$ are elements of $|M_g|$ and $g^{M_g}(\sigma[x_i]) = \sigma[x_i^g]$. Indeed, we have $M_R, \sigma \models \bigwedge_{1 \le i \le n} R_g(x_i, x_i^g)$, and thus $\langle \sigma[x_i], \sigma[x_i^g]\rangle \in R_g^{M_R}$ for every $1 \le i \le n$. Consider an $|M_g|$-assignment σ_g such that $\sigma_g[x_i] = \sigma[x_i]$ for every $1 \le i \le n$. We show that $M_g, \sigma_g' \models \Psi$ for every $\{y_1, \ldots, y_m\}$-variant σ_g' of σ_g. Consequently, $M_g, \sigma_g \models \forall y_1, \ldots, y_m.\Psi$ and hence $M_g \models \Phi$.

Let σ'_g be a $\{y_1, \ldots, y_m\}$-variant σ_g. We prove that $M_g, \sigma'_g \models \Psi$. Consider the $\{y_1, y_1^g, \ldots, y_m, y_m^g\}$-variant σ' of σ defined by $\sigma'[y_i] = \sigma'_g[y_i]$ and $\sigma'[y_i^g] = g^{M_g}(\sigma'_g[y_i])$. It follows from our definitions that $M_R, \sigma' \models \bigwedge_{1 \leq i \leq m} R_g(y_i, y_i^g)$. Therefore, since $M_R, \sigma' \models \bigwedge_{1 \leq i \leq m} R_g(y_i, y_i^g) \rightarrow \Psi'$, we have that $M_R, \sigma' \models \Psi'$. We show that for every atomic Σ_g-formula $p(t_1, \ldots, t_k)$ that occurs in Ψ, $M_g, \sigma'_g \models p(t_1, \ldots, t_k)$ iff $M_R, \sigma' \models p(t'_1, \ldots, t'_k)$, where for every $1 \leq j \leq k$, $t'_j = t_j$ if t_j is a variable, $t'_j = a_i^g$ if $t_j = g(a_i)$ for $a \in \{x, y\}$. By induction on the structure of Ψ, it would easily follow that $M_g, \sigma'_g \models \Psi$ (using the definition of Ψ', and the fact that $M_R, \sigma' \models \Psi'$).

Thus let $p(t_1, \ldots, t_k)$ be an atomic Σ_g-formula that occurs in Ψ, and let $t'_1, \ldots, t'_k$ be defined as above. Using the natural extension of assignments to terms (given interpretations of function symbols) we have that: $\sigma'_g[t_j] = \sigma'[t'_j]$ for every $1 \leq j \leq k$. This is obvious if $t_j = x_i$ or $t_j = y_i$ (since $\sigma'_g[x_i] = \sigma'[x_i]$ for every $1 \leq i \leq n$, and $\sigma'_g[y_i] = \sigma'[y_i]$ for every $1 \leq i \leq m$). But, also if $t_j = g(x_i)$ or $t_j = g(y_i)$, then t'_j is either x_i^g or y_i^g, and we have $\sigma'[x_i^g] = g^{M_g}(\sigma'_g[x_i]) = \sigma'_g[g(x_i)]$ and $\sigma'[y_i^g] = g^{M_g}(\sigma'_g[y_i]) = \sigma'_g[g(y_i)]$. Now, $M_g, \sigma'_g \models p(t_1, \ldots, t_k)$ iff $\langle \sigma'_g[t_1], \ldots, \sigma'_g[t_k] \rangle \in p^{M_g}$, or equivalently iff $\langle \sigma'[t'_1], \ldots, \sigma'[t'_k] \rangle \in p^{M_g}$. Since $\sigma'[t'_j] \in |M_R|$ for every $1 \leq j \leq k$, and $p^{M_g} = p^{M_R} \cap |M_g|^k$, we can equivalently write $\langle \sigma'[t'_1], \ldots, \sigma'[t'_k] \rangle \in p^{M_R}$. This is exactly the condition for $M_R, \sigma' \models p(t'_1, \ldots, t'_k)$. □

Now, note that the obtained formula ψ is a purely relational $\forall^*\exists^*$-formula, and so it is valid in all finite structures iff it is generally valid. It follows that φ is valid (in AE^{AR}) iff ψ is valid (in first-order logic).

Appendix B

Code Examples

SLL: insert

$$h, x, e, i, j \ : \ V$$

$\{P\}$

```
i := h; j := null;
while (i != x & i != null) {I} (
   j := i;
   i := i.n
) ;
j.n := e; e.n := i
```

$\{Q\}$

$$\begin{aligned} P \ &:= \ h \neq \text{null} \wedge h\langle n^+\rangle x \wedge e \neq \text{null} \wedge e\langle n\rangle\text{null} \wedge \neg h\langle n^*\rangle e \\ Q \ &:= \ h \neq \text{null} \wedge h\langle n^*\rangle e \wedge e\langle n\rangle x \\ I \ &:= \ h \neq \text{null} \wedge h\langle n^+\rangle x \wedge e \neq \text{null} \wedge e\langle n\rangle\text{null} \wedge \neg h\langle n^*\rangle e \wedge \\ &\quad \text{ite}(j = \text{null}, i = h, h\langle n^*\rangle j \wedge j\langle n\rangle i) \wedge \\ &\quad i\langle n^*\rangle x \end{aligned}$$

SLL: delete

$$h, i, j \ : \ V$$

$$C \quad : \ V \to \{\mathbf{true}, \mathbf{false}\}$$

$\{P\}$

```
i := h ; j := null ; t := null ;
while (i != null & t = null) {I} (
   if (C(i)) then t := i
   else (
      j := i ; i := i.n
   )
) ;
if (i != null) then (
   if (j = null) then h := i.n
   else ( t := i.n ; j.n := null ; j.n := t )
)
else skip
```

$\{Q\}$

assumptions: $\neg C(\text{null})$, $x, y \neq \text{null}$

$$\begin{aligned}
P &:= h\langle n^* \rangle x \wedge x\langle n^* \rangle y \\
Q &:= x\langle n^* \rangle y \leftrightarrow (x = y \vee y \neq i) \\
I &:= h\langle n^* \rangle x \wedge x\langle n^* \rangle y \wedge \\
&\quad (i \neq \text{null} \to h\langle n^* \rangle i) \wedge \\
&\quad \text{ite}(j = \text{null}, i = h, h\langle n^* \rangle j \wedge j\langle n \rangle i) \wedge \\
&\quad (\forall m(h\langle n^* \rangle m \to \neg i\langle n^* \rangle m \to \neg C(m))) \wedge \\
&\quad (t \neq \text{null} \to C(i))
\end{aligned}$$

SLL: deleteAll

$$h, i, j \ : \ V$$

$\{P\}$

```
i := h;
while $i != null$  $I$  (
   j := i ;
   i := i.n ;
   j.n := null
)
```

$\{Q\}$

$$\begin{aligned} P &:= \forall xy(x\langle n^*\rangle y \leftrightarrow x\langle \underline{n}^*\rangle y) \\ Q &:= \forall x(h\langle \underline{n}^*\rangle x \rightarrow x\langle n\rangle \mathrm{null}) \\ I &:= \forall mw(i\langle n^*\rangle m \rightarrow (m\langle n^*\rangle w \leftrightarrow m\langle \underline{n}^*\rangle w)) \wedge \\ &\quad \forall m(h\langle \underline{n}^*\rangle m \wedge \neg i\langle \underline{n}^*\rangle m \rightarrow m\langle n\rangle \mathrm{null}) \wedge \\ &\quad (i \neq \mathrm{null} \rightarrow h\langle \underline{n}^*\rangle i) \end{aligned}$$

SLL: filter

$\{P\}$

```
i := h ;
j := null ;
while (i != null) {I} (
   if (C(i)) then ( t := i.n ; j.n := null ; j.n := t ) else j := i ;
   i := i.n
)
```

$\{Q\}$

$$
\begin{aligned}
P &:= \underline{n}^* = n^* \wedge \neg C(h) \\
Q &:= \forall y(h\langle n^*\rangle y \leftrightarrow (h\langle \underline{n}^*\rangle y \wedge \neg C(y))) \\
I &:= n^* \subseteq \underline{n}^* \wedge \neg C(h) \wedge \\
&\quad (i \neq \text{null} \rightarrow h\langle n^*\rangle i) \wedge \text{ite}(j = \text{null}, i = h, j\langle n\rangle i \wedge h\langle n^*\rangle j) \wedge \\
&\quad \forall m(m \neq \text{null} \rightarrow \\
&\quad \text{ite}(C(m), h\langle n^*\rangle m \leftrightarrow h\langle \underline{n}^*\rangle m \wedge i\langle n^*\rangle m, \\
&\quad h\langle n^*\rangle m \leftrightarrow h\langle \underline{n}^*\rangle m))
\end{aligned}
$$

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התוצאות פורסמו במאמרים הבאים:

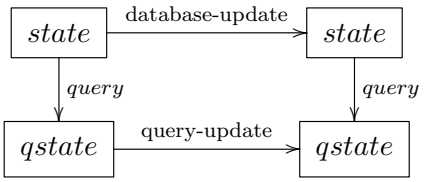
- S. Itzhaky, A. Banerjee, N. Immerman, A. Nanevski, and M. Sagiv. “Effectively-Propositional Reasoning About Reachability in Linked Data Structures” in Computer Aided Verification, 2013, St. Petersburg, Russia, July 2013.
- S. Itzhaky, A. Banerjee, N. Immerman, O. Lahav, A. Nanevski, and M. Sagiv. “Modular Reasoning about Heap Paths via Effectively Propositional Formulas” in 41st ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, San Diego, California, January 2014.
- S. Itzhaky, N. Bjørner, T.Reps, M. Sagiv, and A. Thakur, “Property-Directed Shape Analysis” in Computer Aided Verification 2014, Vienna, Austria, July 2014.

ניתן לשימוש מחדש (code reuse) לטובת הנדסת תוכנה יעילה יותר. שימוש בכלי תכנותי זה גורם לכך שקטע קוד יחיד יכול להיות מבוצע במספר הקשרים שונים, ורצוי מאד שלא יהיה צורך לוודא את נכונותו של קטע הקוד בכל אחד ואחד מן ההקשרים שבהם הוא מופיע בנפרד, שכן זה יגרום לניתוח להיות לא סקלבילי ולא ניתן יהיה לוודא בשיטה זו תכניות המכילות מספר רב של פרוצדורות. האתגר הוא היכולת להביע את העדכון באופן שיתמצת את הפעולה, או האפקט, שמבצעת הפרוצדורה על הערמה בהקשר כלשהו, כאשר חלק מאברי הערמה נגישים לפרוצדורה והחלק האחר אינו נגיש, ולפיכך בהכרח אינו יכול להשתנות. אפשר לראות זאת כמקרה של בעיית מסגרת (frame problem).

בתזה מוצגת משפחת לוגיקות תחת השם **לוגיקות ישיגות (reachability logics)**. ניתנת הגדרה פורמלית של מקטעים לוגיים שנמצאו שימושיים לצורך היסק שיטתי אודות תכניות הכוללות מבנים עם מצביעים. כשיטה עיקרית, הסמנטיקה של לוגיקות אלה מוטמעת בלוגיקה מסדר ראשון על-מנת לאפשר שימוש בכלי הוכחה אוטומטיים היודעים לקבל קלט בשפה זו. שימוש זה מהווה, אמנם, מגבלה על יכולת ההבעה של אותן לוגיקות, אולם היתרונות הנובעים מהשימוש בכלים אוטומטיים על-פני הצורך לספק הוכחות באופן ידני, אפילו כשמדובר בתכניות קטנות שנראות מובנות-מאליהן, הופכים את השימוש בהם לכדאי בכל עת שניתן.

אבל חילופי כמתים הינם אסורים. הסיבה העיקרית לכך היא ששמורות לולאה מופיעות בשני הצדדים של קשר גרירה כאשר בונים את תנאי האימות; לפיכך נדרש כי שפת ההכרזות תהיה סגורה תחת שלילה, בעוד שעבור תנאי האימות הסופי לא קיימת דרישה כזו.

הפנייה לסיבוכיות תיאורית נובעת מכך שלאחרונה נעשה בה שימוש בבעיות עדכון של בסיסי נתונים. לבעיה זו יש מקבילה חביבה לעדכוני הישיגות בערמה שתכניות שאנו בוחנים. בבעיית view-update ניתן להראות, שסיבוכיות הנוסחה המשמשת לעדכון תוצאות שאילתה על-פני שינויים המתרחשים בבסיס הנתונים היא נמוכה יותר מאשר הנוסחה המשמשת לחישוב השאילתה מההתחלה. אכן, לצורך חישוב הישיגות יש צורך בנוסחה עם סגור טרנזיטיבי, בעוד שנוסחאות עדכון ניתן לבטא במונחים מסדר ראשון בלבד. אנחנו מנצלים את העובדה שהסיבוכיות של התאמת הישיגות ה"ישנה" למצב ה "חדש" נמוכה מהסיבוכיות של חישוב ישיגות. הפתרון שאנו מאמצים דומה לשימוש באלגוריתמיים דינמיים על גרפים עבור פתרון בעיות view-update, כאשר מסלולים מכוונים בין צמתים משתנים תוך כדי מחיקה או הוספה של קשת; אלא שהפתרון שלנו מכוון לטיפול בתכניות מחשב המבצעות את העדכונים באמצעות מצביעים היוצרים מבני נתונים מקושרים.



פן נוסף המסבך היסק תכנותי, במיוחד כשמדובר במצבים מורכבים כגון אלה שבהם נתקלים כאשר קיימים מבני נתונים המכילים מצביעים, הוא פרוצדורות. תכניות מחשב מורכבות בדרך-כלל ממספר תת-תכניות, וזאת על-מנת שהקוד יהיה קריא יותר ויהיה

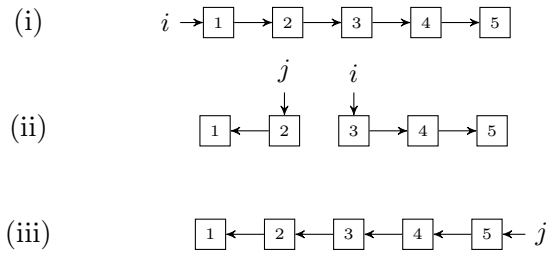
בתזה זו נתבסס בעיקר על הפיתוחים של לוגיקת הור (Hoare logic) והרחבותיה, המכונות בדרך-כלל בשם הכולל "אימות נוסח-הור". לוגיקת הור היא מערכת הוכחה לעריכת היסקים אודות תכניות ומפרטיהן הנתונים כהכרזות (assertions) - בעיקר תנאי-קדם (pre-condition) ותנאי-בתר (post-condition) - מובעים בשפה לוגית לבחירתנו. בעוד שקיימות מערכות הוכחה שלמות עבור שפות תכנות אימפרטיביות (דוגמה אחת למערכת כזו מוצגת ברקע לעבודה), בעיית חיפוש ההוכחה היא המכשול העיקרי בדרך למימוש מערכת אוטומטית לאימות תכניות ולהוכחת תכונותיהן. זה נכון גם עבור תכניות קטנות ביותר, כיוון שמרחב ההוכחות הוא עצום - במיוחד כאשר התכנית מכילה לולאות. לולאה בתכנית עשויה לגרום לקטע קוד לרוץ מספר כלשהו של פעמים, כך שמספר המצבים השונים שבהם התכנית נתקלת במהלך ריצתה אינו פרופורציוני בגודל התכנית. באופן גס ניתן לומר שטיעונים העוסקים בקבוצות של מצבים מציבים בעיית היסק מסדר גבוה.

שתי אבחנות מרכזיות נמצאות בבסיס השיטה.

(א) בתכניות המנהלות רשימות מקושרות ודו-מקושרות, ניתן להביע את המצביע next במונחים של מסלולים, או במילים אחרות יחס ישיגות בין איברי הרשימה. זה מאפשר שימוש נוח בתוצאות חדשות מתורת הסיבוכיות התיאורית (descriptive complexity): אפשר לתחזק את יחס הישיגות בהתאם לשינויים הנערכים בערמה באופן שאינו מאבד מידע ולפיכך הוא מדויק. זאת ועוד, האקסיומות הנדרשות לצורך העדכון הינן ללא כמתים.

(ב) על-מנת לטפל בתכניות שסורקות את הערמה, ניתן לייצר תנאי אימות בלוגיקה מסדר ראשון שצורתם תהיה $\forall^*\exists^*\phi$ כך שניתן יהיה לוודא את אמיתותם בעזרת כלים לבדיקת ספיקות (SAT - כפי שיוסבר בהמשך). למתכנת יותר לכתוב הכרזות בשפה מצומצמת יותר של לוגיקה מסדר ראשון, שבה ניתן להשתמש בסגור טרנזיטיבי של next

מסלול-next בין j ל i בשום שלב של הריצה, שכן אם קיים כזה מסלול, אז הוספת הקשת i→j יוצרת מעגל התייחסויות בערמה.



אנו ניגשים לבעיה זו על-ידי בניה קפדנית של שמורות לולאה מתאימות עבור תכניות איטרטיביות, ומפרטים שלמים ומדויקים של פרוצדורות עבור תכניות רקורסיביות. אם נתבונן בריצה אופיינית של התכנית reverse, ניתן להבחין בתכונה חשובה: המשתנים i ו j מצביעים בכל עת על תחילתם של שני מקטעי רשימה **זרים**. כל אחד מהמקטעים עשוי להיות ריק (כמו במצבים (i) ו (iii)) אך לעולם לא יהיו להם איברים משותפים. מסתבר שתכונה זו היא הכרחית עבור הוכחת הנכונות של התכנית להיפוך רשימה. ניסוח הטענה בלוגיקה מורכב יותר מאשר בדוגמאות הקודמות, עם תנאי החפיפה הפשוטים. כדי לאפשר ניסוחים כאלה אנו מגדירים לוגיקות ישיגות (reachability logics) ומספקים שיטות היסק להוכחת נכונות של גרירות ונביעות. בגישה זו, נכתוב את שמורת הלולאה כך:

$$\forall \alpha : \mathrm{i}\langle \mathrm{next}^* \rangle \alpha \wedge \mathrm{j}\langle \mathrm{next}^* \rangle \alpha \rightarrow \alpha = \mathrm{null}$$

בעוד שריינולדס גרס שגישה מעין זו לעולם לא תהיה סקלבילית, אנו מראים שעבור מקרים רבים המתרחשים בתכניות מציאותיות השמורות ניתנות בהחלט לטיפול ושיטות היסק אוטומטיות מתאפשרות בזמן ריצה ודרישות זכרון סבירים. על-מנת לתמוך בתכניות ההולכות וגדלות אנחנו ממשיכים לפתח לוגיקות להיסק מודולרי. בדוגמה הספציפית של ריינולדס נדון בפירוט בפרק 3.

מכאן קל לזהות את הפגם: במקרה ש x=y (חפיפה) המפרט סותר את התכנית. זאת מכיוון שהמפרט או דקלרטיבי (הצהרתי), כך שסדר המשוואות הינו חסר-משמעות ולמעשה שני השוויונות מציינים את אותה התכונה עצמה: שתא הזכרון שנמצא בכתובת המאוחסנת במשתנים x ו y גדל בערכו ב 1. הסמנטיקה של התכנית, לעומת זאת, היא אימפרטיבית, והעובדה שההשמה חוזרת על עצמה היא, כמובן, משמעותית - הערך יגדל ב 2. שימוש בכל סוג של סמנטיקה פורמלית מספק דרך שיטתית לגילוי חוסר-התאמה מסוג זה.

צירופים מורכבים של מצביעים ופעולות על כתובות מובילים לטיעונים מסובכים וקשים יותר לביצוע. תכנית הדוגמה שלהלן שימשה את ריינולדס באחד המאמרים הראשונים העוסקים ב Separation logic. התכנית הופכת את סדר האיברים ברשימה מקושרת הניתנת לה כקלט; המשתנה i מכיל, במצב ההתחלתי, מצביע לראש הרשימה; לכל איבר ברשימה קיים שדה בשם next המחזיק מצביע לאיבר הבא ברשימה (האיבר האחרון יאחסן null בשדה זה). התכנית בונה מהאיברים רשימה בעלת מבנה זהה, אך עם סדר איברים ההפוך מזה שבקלט. פעולת ההיפוך נעשית "במקום", כך שרשימת הקלט משוכתבת על-ידי הפלט.

$$j := \text{null};\ \text{while } i \neq \text{null do}$$

$$(k := i.\text{next} ;\ i.\text{next} := j ;\ j := i ;\ i := k)$$

ריינולדס זיהה בעיה יסודית בהיסקים העוסקים בתכניות שסורקות מבני נתונים רקורסיביים שכאלה: המצביע i המשמש כאיטרטור מקודם בכל צעד, ומספר הצעדים אינו חסום - הוא תלוי בגודל הקלט. לפיכך ישנה תמיד סכנה שערך שנכתב בצעד מסוים של הלולאה ישוכתב באחד הצעדים הבאים, ובכך יפר את התכונה הנדרשת מהתכנית. למשל, כדי לוודא שבתכנית הדוגמה לא נוצר מעגל ברשימה, חייבים להוכיח שאין

עקב הפופולריות הרבה שצברה, מצביעים הם כי עבודה בסיסי כמו ארבע פעולות החשבון באריתמטיקה ומבני הבקרה של התכנית. הם משלבים כושר ביטוי גבוה עם יעילות זמן-ריצה בביצוע פעולות זיכרון פרימיטיביות. Seperation logic, לוגיקה שפותחה בעיקר על-ידי ריינולדס ואו'הרן, הותאמה במיוחד לענות על אתגרים אלו של התכנות. אתגרים עולים, ככלל, עקב הנוכחות של חפיפה (aliasing) בתכניות עם מצביעים: זהו מצב שבו שני מצביעים מכילים את אותה כתובת, ומשום-כך שינוי בנתון המאוחסן בכתובת זו הופך זמין מיד בעת ובעונה אחת בשני מקומות בתכנית. ניתן להדגים זאת על-ידי עיון בתכנית הבאה (בשפת C):

```
void go_up(int *x, int *y) {
  (*x)++; (*y)++;
}
```

כוונת המתכנת הייתה לקדם את שני המונים המוצבעים על-ידי x ו y. ואולם, במקרה קצה שבו הפרמטרים x ו y מכילים מצביע לאותה כתובת (הם חופפים), התוצאה תהיה לא צפויה: הגדלה של מונה יחיד ב 2. אם המתכנת לא התכונן לתרחיש כזה, הוא עלול להוביל לשגיאות זמן-ריצה ולבאגים עדינים וקשים לאיתור. ניסיון להוכיח את נכונות השגרה באופן פורמלי יביא מיד לניסוח מפרט אפשרי עבורה, למשל זה המאופין על-ידי המשוואות:

$[x] = \underline{[x]} + 1$

$[y] = \underline{[y]} + 1$

כאשר בסימון המופיע כאן, כמקובל באפיונים של תכניות המבצעות שינוי הרסני, מביע $\underline{x}$ את ערך הקלט של משתנה התכנית x, במצב שלפני השינוי, ואילו x מביע את ערך הפלט שלו אחרי השינוי. הסוגריים המרובעים מציינים התייחסות לערכים המאוחסנים בכתובת המוצבעת על ידי המשתנה שבמשוואה, במקום לכתובות עצמן, שאינן משתנות.

תקציר

תזה זו מציגה אמצעים לצורך הוכחה אוטומטית של נכונות תכניות מחשב העושות שימוש נרחב במצביעים. אלה כוללות תכניות שמנהלות מבני נתונים מקושרים, כגון רשימות מקושרות, רשימות דו-מקושרות, רשימות מקוננות ועצים מהופכים. לצורך המיכון משתמשים בכלים שהיו לסטנדרט בתעשיה ושיעילותם הוכחה בפועל. התשתית הלוגית המוצעת מבוססת על רדוקציה של בעיית האימות לסדרה של שאילתות לוגיות. אלה מיוצרות אוטומטית ונכללות בקבוצה כריעה, לפיכך תשובה מוחלטת ("כן" או "לא") מובטחת בכל מקרה.

בעיית אימות התוכנה קיימת מאז תחילת קיומה של התוכנה עצמה. החוקרים פלויד והור יזמו תשתית לוגית לעריכת הוכחות נכונות של תכניות מחשב אל-מול מפרט פורמלי; הן הוכחת נכונות מלאה, אשר כוללת הוכחה שהתכנית הנדונה עוצרת תוך זמן סופי על מחלקת הקלט המיועדת לה, בד-בבד עם התנהגות התואמת את המפרט, והן הסוג הנפוץ יותר של הוכחת נכונות חלקית, אשר מקלה את הדרישה ומאפשרת ריצות לא עוצרות של התכנית. מאמץ מתמשך למיכון עריכת הוכחות אלה ממשיך להתקיים מאז ועד היום. הזוג קוסו הביאו לפיתוחו של תחום הפירוש המופשט (abstract interpretation), המספק אוסף נרחב של טכניקות ושיטות-עזר עבור ניתוח תוכניות מסוגים שונים. שיטות אלה נמצאות בשימוש יומיומי במהדרים מודרניים הודות לאלגנטיות המתמטית שלהן, קלות המימוש, והביצועים הטובים שהן מספקות. באופן טבעי, קיים תמיד מתח בין כמות המשאבים הנדרשת לניתוח לבין דיוק התוצאות, מה שמוביל לכך שרוב האנליזות הנמצאות בשימוש כיום מספקות תוצאות מקורבות.

מתוך מרחב הבעיות הקשורות באימות תוכנה, מעניינת במיוחד קבוצה של בעיות הקשורות לתכניות מחשב העושות שימוש נרחב במצביעים. בשפת C ובנגזרותיה שנוצרו

אלגוריתם עידון איטרטיבי עבור הסקה אוטומטית של שמורות אינדוקטיביות מעל קבוצה של פרדיקטים המשמשים להפשטה (אבסטרקציה) של מרחב המצבים. האלגוריתם בונה בהדרגה אוסף של הפשטות אשר חוסמות מלמעלה את קבוצת המצבים הישיגים עד שמתקבלת שמורה שהינה חזקה מספיק בשביל להוכיח את התכונה הנדרשת. גישה זו ידועה בשם "property-directed reachability". הראינו שהמימוש שלנו מצליח לייצר שמורות נכונות עבור קבוצה של מקרי-בוחן עם מפרטים מתאימים.

תמצית

תזה זו מנסחת שיטה חדשה עבור בעיות אימות של תכניות המנהלות מבני נתונים מסוג רשימה מקושרת ומבנים דומים המשתמשים בחוליות כמו עצים הפוכים. השיטה עושה שימוש בתת-קבוצה מוגבלת של לוגיקה מסדר ראשון שהינה כריעה, ועם-זאת אפקטיבית דיה כדי לתמוך בהיסק אודות מסלולים של קישורי הצבעות בערמה הדינמית של התכנית. תכונות כאלה הכרחיות כדי להוכיח עצירה, תקינות של מבנה הנתונים, ותכונות בטיחות נוספות. ליבת התזה היא הצרנה אקסיומטית שלמה של סגור טרנזיטיבי דטרמיניסטי בלוגיקה מסדר ראשון, בנוסחאות פסוקיות-למעשה (EPR), כך שניתן לרתום תכניות קיימות לפתרון SAT לצורך הוכחת תקפות, או לצורך ייצור של דוגמת-נגד קונקרטית עבור מקרים שתנאי האימות מופר. כיוון שהפתרון שלם לוגית ומוכל כולו במקטע כריע של לוגיקה מסדר ראשון, אחת מתוצאות אלה מובטחת - השגרה לעולם אינה מתבדרת ואינה מחזירה תוצאה בלתי-מדויקת.

אנו מציגים בנוסף טכניקות להיסק מודולרי. בתכניות המכילות פרוצדורות, קיימת התופעה של "אפקט גלובלי": תת-שגרה שמבצעת שינוי בחלק קטן של הערמה עשויה להשפיע על תכונות ישיגות (דרך קישורי הצבעה) במקומים שונים בערמה, מכיוון שהפעולות שלה מנתקות מסלולים קיימים ויוצרות מסלולים חדשים. לפיכך ישנן הגבלות מסויימות על מה מותר ומה אסור לשגרה הקוראת ולשגרה הנקראת לבצע. נעשה שימוש בכלל אדפטציה מתאים כדי לשלב את השינויים שעשתה תת-השגרה הנקראת בתוך מרחב הערמה של השגרה שקראה לה.

בשתי האנליזות הללו, המשתמש נדרש לספק שמורה אינדוקטיבית מתאימה עבור לולאות ועבור פרוצדורות רקורסיביות. השמורות אינן טריוויאליות כלל ועלולות להיות מסובכות יותר מאשר המפרט של התכנית כולה. כדי להתגבר על הבעיה הזו, יישמנו

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היסק אוטומטי בתכניות עם מצביעים על-ידי לוגיקות כריעות

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