In Lecture 10 we introduced the notion of a graph polynomial.

- The chromatic polynomial was introduced and many facts about it were presented.
- We proved that there are many, MANY, graph polynomials.
- We have listed many explicit examples: Variations on colorings and others.

**Homework:** Reread the slides of Lecture 10!
Lecture 11
Comparing graph polynomials

- Distinctive power of graph polynomials
- $P$-equivalence and complete graph polynomials
- Reducibility via coefficients
Comparing graph parameters and graph polynomials

Jointly prepared with E.V. Ravve
Graph parameters and graph polynomials

Let $\mathcal{R}$ be a (possibly ordered) ring or a field.

For a set of indeterminates $\bar{X}$ we denote by $\mathcal{R}[^{\bar{X}}]$ the polynomial ring over $\mathcal{R}$.

A graph parameter $p$ is a function from the class of all finite graphs $\text{Graphs}$ into $\mathcal{R}$ which is invariant under graph isomorphism.

A graph polynomial $p$ is a function from the class of all finite graphs $\text{Graphs}$ into $\mathcal{R}[\bar{X}]$ which is invariant under graph isomorphism.

Remark: In most situations in the literature $\mathcal{R}$ is $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$. The choice of the underlying ring or field may depend on the way we want to represent the graph parameter or graph polynomial.

For the graph parameter $d_{\text{max}}(G)$, the maximal degree of its vertices, $\mathbb{Z}$ suffices, but for $d_{\text{average}}(G)$, the average degree of its vertices, $\mathbb{Q}$ is needed.
Equivalence of graph polynomials, I

Let $\mathcal{C}$ be a graph property. Let $P(G, \overline{X})$ and let $Q(G, \overline{Y})$ be two graph polynomials.

**Definition 1**

We say that $Q$ determines $P$ over $\mathcal{C}$, or $Q$ is at least as distinctive than $P$ over $\mathcal{C}$, and write $P \preceq_{d.p.}^{\mathcal{C}} Q$ if for all graphs $G_1$ and $G_2$ in $\mathcal{C}$,

$$Q(G_1) = Q(G_2) \text{ implies that } P(G_1) = Q(P_2).$$

- If $\mathcal{C}$ consists of all graphs, we omit $\mathcal{C}$.

- The definition also applies to graph parameters $P(G), Q(G) \in \mathbb{Z}$.

$P$ and $Q$ are $d.p.$-equivalent over $\mathcal{C}$, and write $P \sim_{d.p.}^{\mathcal{C}} Q$, iff $P \preceq_{d.p.}^{\mathcal{C}} Q$ and $Q \preceq_{d.p.}^{\mathcal{C}} P$. 
Examples of $P \preceq^C_{d.p.} Q$

(i) (DKT, 3.2.1) The chromatic polynomial $\chi(G, X)$ determines the graph parameters $|V(G)|$, $|E(G)|$, $\chi(G)$, $k(G)$, $b(G)$, $g(G)$, etc.

(ii) $d_{max}$ and $d_{average}$ are d.p-incomparable.

(iii) The Tutte polynomial $T(G, X, Y)$ determines $\chi(G, X)$ on connected graphs, but not on all graphs.

(iv) Assume $P(G; X), Q(G; X), U(G, X)$ are three polynomials and $P(G, X) = U(G, X) \cdot Q(G, X)$.
Let $C_U$ be a class of graphs such that for all $G_1, G_2 \in C_U$ we have $U(G_1, X) = U(G_2 : X)$. Then $P \preceq_{d.p.}^C Q$.

(v) Let $F$ be the class of forests. For the characteristic polynomial $\text{char}(G, \lambda)$ and the matching polynomial $\text{dm}(G, \lambda)$ and we have

$$\text{char} \sim_{d.p.}^F \text{dm}.$$
Adjoint polynomials

Let \( P(G, \lambda) \) be a graph polynomial. We denote by \( \bar{G} \) the complement graph of \( G \).

The adjoint polynomial \( \bar{P}(G, \lambda) \) is the polynomial defined by
\[
\bar{P}(G, \lambda) \overset{\text{def}}{=} P(\bar{G}, \lambda)
\]

- **Exercise:** \( P \preceq_{\text{d.p.}} \bar{P} \) iff \( \bar{P} \preceq_{\text{d.p.}} P \)

- For the Tutte polynomial \( T(G, X, Y) \) and \( \bar{E}_n = K_n \) we have
  (i) \( T(E_m) = T(E_n) = 1 \) for all \( n \in \mathbb{N} \).
  (ii) \( T(K_m) \neq T(K_n) \) for \( m \neq n \).
  (iii) Hence the Tutte polynomial and its adjoint are not \( \text{d.p.-comparable} \).
$P$-unique and $P$-equivalent graphs

**Definition 2** Let $P = P(G; \bar{X})$ a graph polynomial and $C$ a class of graphs.

(i) Two graphs $G_1$ and $G_2$ are $P$-equivalent for $C$ if $P(G; \bar{X}) = P(G_1; \bar{X})$.

(ii) A graph $G \in C$ is $P$-unique for $C$ if for any other graph $G_1 \in C$ with $P(G; \bar{X}) = P(G_1; \bar{X})$ the graph $G_1$ is isomorphic to $G$.

(iii) $P$ is complete for $C$ if every graph $G \in C$ is $P(G; \bar{X})$-unique for $C$.

If $C$ consists of all graphs we omit $C$.

**Proposition 3** Let $P$ and $Q$ be graph polynomials such that $P \preceq_{d.p.} Q$.

(i) If $G_1$ and $G_2$ are $Q$-equivalent for $C$ then they are also $P$-equivalent for $C$.

(ii) If $G$ is $P$-unique for $C$ then $G$ is $Q$-unique for $C$.

(iii) If $P$ is complete for $C$ then $Q$ is complete for $C$. 

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Complete graph polynomials

Are there complete graph polynomials?

The following is a graph-complete graph invariant.

- Let $X_{i,j}$ and $Y$ be indeterminates.
  For a graph $\langle V, E \rangle$ with $V = [n]$ we put

$$Compl(G, Y, \bar{X}) = Y^{|V|} \cdot \left( \sum_{\sigma \in \mathcal{S}_n} \prod_{(i,j) \in E} X_{\sigma(i), \sigma(j)} \right)$$

Here $\mathcal{S}_n$ is the permutation group of $[n]$.

**Challenge:** Find a polynomial in a constant finite number of indeterminates which is a graph-complete graph invariant.
An “unnatural” graph-complete invariant

Let \( g : \mathcal{G} \rightarrow \mathbb{N} \) be a Gödel numbering for labeled graphs of the form \( G = \langle [n], E, <_{\text{nat}} \rangle \).

We define a graph polynomial using \( g \):

\[
\Gamma(G, X) = \sum_{H \cong G} X^{g(H)}
\]

Clearly this is a graph invariant.

But it is “obviously unnatural”!

Can we make precise what a natural graph polynomial should be?
\(\chi\)-equivalent graphs (from [DKT, chapter 5])

(i) The graphs \(E_n, K_n\) and \(K_{n,n}\) are \(\chi\)-unique for \(n \geq 1\).

(ii) The graphs \(C_n\) are \(\chi\)-unique for \(n \geq 3\), \(C_i = K_i\) for \(i \leq 2\).

(iii) Any two trees on \(n\) vertices are \(\chi\)-equivalent.

In [DKT, chapter 5] many pairs of \(\chi\)-equivalent graphs are constructed using a method due to R.C. Read (1987) and G.L. Chia (1988).

Research project:

Study \(P\)-equivalence for the various generalized colorings of Lecture 10.
**char**-equivalent graphs

From M. Noy, Graphs determined by polynomial invariants (2003)

Let $\text{char}(G, x) = \det(x \cdot 1 - A_G)$ be the characteristic polynomial of $G$ with adjacency matrix $A_G$.

(i) The graphs $K_{n,n}$ are $\text{char}$-unique.

(ii) The line graphs $L(K_n)$ are $\text{char}$-unique for $n \neq 8$.
    For $n = 8$ there are three exceptions.

(iii) The line graphs $L(K_{n,n})$ are $\text{char}$-unique for $n \neq 4$.
    For $n = 4$ there is one exception.
The two matching polynomials

Recall, for $G = (V, E)$ with $|V| = n$,

$$dm(G, x) = \sum_r (-1)^r m_r(G) x^{n-2r}$$

be the (defect) matching polynomial and

$$gm(G, x) = \sum_r m_r(G) x^r$$

the (generating) matching polynomial.

We have

$$dm(G, x) = x^n gm(G; (-x)^{-2})$$
Graphs equivalent for matching polynomials.

From M. Noy, Graphs determined by polynomial invariants (2003)

• For every graph $G$ we have $gm(G, x) = gm(G \sqcup E_n, x)$ but $dm(G, x) \neq dm(G \sqcup E_n, x)$.

$$dm(P_2, x) = x^2 - 1 \text{ and } dm(P_2 \sqcup E_k, x) = x^3 - x,$$
but $gm(P_2, x) = x^2 - 1 \text{ and } gm(P_2 \sqcup E_k, x) = x^2 - 1$

• $|V(G)| \preceq_{d.p.} dm$, and therefore $gm \preceq_{d.p.} dm$.
In other words $gm$ is strictly less expressive than $dm$.

• $gm \sim_{d.p.} dm$ on graphs of a fixed number of vertices.

• The graphs $K_{n,n}$ are $dm$-unique.
  Are they also $gm$-unique?

Research project:

Study $dm$-equivalence and $gm$-equivalence of graphs further.
$T$-unique graphs

From A. de Mier and M. Noy, On Graphs determined by the Tutte polynomial (2004)

For a graph $G = (V, E)$ and $A \subseteq E$ we denote by $G[A] = (V, A)$ the spanning subgraph generated by $A$. We set $r(A) = |V| - k(G[A])$ and $n(A) = |A| - r(A)$.

The Tutte polynomial is defined by

$$T(G; X, Y) = \sum_{A \subseteq E} (X - 1)^{r(E) - r(A)} (Y - 1)^{n(A)}$$

(i) Recall that $\chi \preceq_d p. T$ on connected graphs. Hence the graphs $K_{n,n}$ are $T$-unique.

(ii) The wheels $W_n$ are $T$-unique for all $n \in \mathbb{N}$.

Wheels are $\chi$-unique for $W_{2n}$, $W_5$ and $W_7$ are not. In general it is not known (?) whether $W_{2n+1}$ is $\chi$-unique.

(iii) The ladders $L_n$ are $T$-unique for all $n \geq 3$.

They are only known to be $\chi$-unique for small values of $n$. 

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Bollobas-Pebody-Riordan Conjecture:

Almost all graphs are $T$-unique and even $\chi$-unique

Let us make it more precise:

Let $TU$ ($\chi U$) be the graph property:
$G \in TU$ ($G \in \chi U$) iff $G$ is $T$-unique ($\chi$-unique),
and $TU(n)$ ($\chi U(n)$) be the density function of $TU$ ($\chi U$).

The conjecture for the Tutte polynomial now is

$$\lim_{n \to \infty} \frac{TU(n)}{\binom{n}{2}} = 1$$

Similar for $\chi(G, \lambda)$.

Is $TU$ ($\chi U$) definable in some logic with a 0–1-law?

B. Bollobás, L. Pebody and O. Riordan, Contraction-Deletion Invariants for Graphs,
Almost complete graph invariants

A graph polynomial $P$ is almost complete, if almost all graphs are $P$-unique.

Research problems:

- Study the definability of the graph property $G$ is $P$ unique for various graph polynomials $P$.

- Find natural graph polynomials which are almost complete.

- In particular, is the signed Tutte polynomial $T_{\text{signed}}$ almost complete for signed graphs.

A positive answer would be interesting for knot theorists: $T_{\text{signed}}$ is intimately related to the Jones polynomial of knot theory.
Comparison of graph polynomials by coefficients
Coefficients of graph polynomials, I: The univariate case

We denote by $\mathbb{Z}^{<\omega}$ the finite sequences of elements of $\mathbb{Z}$.

Let $P(G, X) \in \mathbb{Z}[X]$ and $P(G, X) = \sum_{i=0}^{d(G)} a_i(G) \cdot X^i$ with $a(G)_{d(G)} \neq 0$.

We denote by $cP(G, X)$ the finite sequence $(a_i(G))_{i \leq d(G)} \in \mathbb{Z}^{<\omega}$.

$cP(G, X)$ are the (standard) coefficients of $P(G, X)$, and $d(G)$ is its degree.

$c$ is a one-one and onto function $c : \mathbb{Z}[X] \xrightarrow{c} \mathbb{Z}^{<\omega}$.

Instead of looking at graph polynomials $P : \text{Graphs} \xrightarrow{P} \mathbb{Z}[X]$, we can look at the function $cP : \text{Graphs} \longrightarrow \mathbb{Z}^{<\omega}$ defined by

$cP : \text{Graphs} \xrightarrow{P} \mathbb{Z}[X] \xrightarrow{c} \mathbb{Z}^{<\omega}$

**Lemma 4**

For all graphs $G_1, G_2$, we have that $P(G_1) = P(G_2)$ iff $cP(G_1) = cP(G_2)$.
Other representations of graph polynomials

Our definition of $cP$ uses the **power form of $P$**.

We could have used also **factorial form** or **binomial form** of $P$.

- $cP$ denotes the coefficients of $P$ in power form.
- $c_1P$ denotes the coefficients of $P$ in factorial form.
- $c_2P$ denotes the coefficients of $P$ in binomial form.

We note that there are simple algorithms to pass from one representation to another.
Equivalence of graph polynomials, II

Let $\mathcal{C}$ be a graph property.
Let $P(G, \bar{X})$ and Let $Q(G, \bar{Y})$ be two graph polynomials.

**Definition 5**
We say that $Q$ determines coefficient-wise $P$ over $\mathcal{C}$ and write $P \preceq_{\text{coeff}}^{\mathcal{C}} Q$ if there is a function $F : \mathbb{Z}^<\omega \to \mathbb{Z}^<\omega$ such that for all graphs $G \in \mathcal{C}$

$$F(cQ(G)) = cP(G)$$

$P$ and $Q$ are coefficient-equivalent over $\mathcal{C}$, and write $P \sim_{\text{coeff}}^{\mathcal{C}} Q$, iff $P \preceq_{\text{coeff}}^{\mathcal{C}} Q$ and $Q \preceq_{\text{coeff}}^{\mathcal{C}} P$

- If $\mathcal{C}$ consists of all graphs, we omit $\mathcal{C}$.
- The definition also applies to graph parameters $P(G), Q(G) \in \mathbb{Z}$.
- Our definition is invariant under the choice of representations $cP, c_1P$ or $c_2P$. 
An example: $F$ can be arbitrarily complex

Let $P(G, \lambda) = \sum_i a_i(G) \lambda^i$.

Let $P_{\text{exp}}(G, \lambda) = \sum_i 2^{a_i(G)} \lambda^i$,

and for $g : \mathbb{N} \rightarrow \mathbb{N}$ one-one and onto let $P_g(G, \lambda) = \sum_i a_i(G) \lambda^{g(i)}$.

Clearly,

$$P \sim_{\text{coeff}} P_g \sim_{\text{coeff}} P_{\text{exp}}$$

- If $g$ is not computable, then $F$ showing that $P \sim_{\text{coeff}} P_g$ cannot be computable in the Turing model of computation.
- Furthermore, $F$ showing that $P \sim_{\text{coeff}} P_{\text{exp}}$ cannot be computable in the Blum-Shub-Smale model of computation.
Theorem 6  \( P \preceq^C_{coeff} Q \) iff \( P \preceq^C_{d.p.} Q \)
Proof: $P \preceq^C_{coef} Q$ implies $P \preceq^C_{d.p.} Q$.

Assume there is a function $F : \mathbb{Z}^{<\omega} \rightarrow \mathbb{Z}^{<\omega}$ such that for all graphs $G \in \mathcal{C}$ we have $F(cQ(G)) = cP(G)$.

Now let $G_1, G_2 \in \mathcal{C}$ such that $Q(G_1) = Q(G_2)$.

By Lemma 4 we have $cQ(G_1) = cQ(G_2)$.

Hence $F(cQ(G_1)) = F(cQ(G_2))$.

Since for all $G \in \mathcal{C}$ we have $F(cQ(G)) = cP(G)$, we get $cP(G_1) = cP(G_2)$ and, using Lemma 4 again, we have $P(G_1) = P(G_2)$. 
Proof: \( P \preceq_{d.p}^{C} Q \) implies \( P \preceq_{\text{coeff}}^{C} Q \).

We use the well-ordering principle which equivalent to axiom of choice.

Let \( \{F_{\alpha} : \alpha < \beta\} \) be a well-ordering of all the functions \( F : \mathbb{Z}^{<\omega} \rightarrow \mathbb{Z}^{<\omega} \).

For \( G \in \mathcal{C} \), let \( \gamma(G) < \beta \) be the smallest ordinal such that \( F_{\gamma(G)}(cQ(G)) = cP(G) \).

Now given \( P(G, X) \preceq_{d.p} Q(G, X) \), we define a function \( F_{P,Q} : \mathbb{Z}^{<\omega} \rightarrow \mathbb{Z}^{<\omega} \) as follows:

\[
F_{P,Q}(cQ(G)) = \begin{cases} 
F_{\gamma(G)}(cQ(G)) & \text{if } G \in \mathcal{C} \\
0 & \text{else}
\end{cases}
\]

Using Lemma 4 and \( P(G, X) \preceq_{d.p} Q(G, X) \), this indeed defines a function.

Finally, as \( F_{\gamma(G)}(cQ(G)) = F_{\gamma(G)}(cP(G)) \), we get

\[
F_{P,Q}(cQ(G)) = cP(G)
\]

Q.E.D.
A proof without well-ordering (suggested by Ofer David)

Let $S$ be a set of finite graphs and $s \in \mathbb{Z}^{<\omega}$.
For a graph polynomial $P$ we define:

$$P[S] = \{s \in \mathbb{Z}^{<\omega} : cP(G) = s \text{ for some } G \in S\} \text{ and } P^{-1}(s) = \{G : cP(G) = s\}.$$ 

Now assume $P(G, X) \preceq d.p. Q(G, X)$.

If $Q^{-1}(s) \neq \emptyset$, then for every $G_1, G_2 \in Q^{-1}(s)$ we have $cQ(G_1) = cQ(G_2)$, and therefore $cP(G_1) = cP(G_2)$.

Hence $P[Q^{-1}(s)] = \{t_s\}$ for some $t_s \in \mathbb{Z}^{<\omega}$.

Now we define

$$F_{P, Q}(s) = \begin{cases} t_s & \text{if } Q^{-1}(s) \neq \emptyset \\ s & \text{else} \end{cases}$$

Q.E.D.
Example, I: The two matching polynomials

\[ dm(G, x) = \sum_r (-1)^r m_r(G)x^{n-2r} \]
\[ gm(G, x) = \sum_r m_r(G)x^r \]

We have \( dm(G; x) = x^n gm(G; (-x)^{-2}) \) where \( n = |V| \).

- The degree of \( dm \) is \( n \)
- If \( m_r(G) \neq 0 \) the \( n - 2r > 0 \).
- Hence

\[ \frac{dm(G; x)}{X^n} \]

is a polynomial, and we can compute the coefficients of \( gm \) from the coefficients of \( dm \).

- We cannot compute the coefficients of \( dm \) from \( gm \) without knowing the value of \( |V| = n \).
Example II: The Tutte polynomial and the chromatic polynomial

The Tutte polynomial and the chromatic polynomial are related by the formula

$$\chi(G, X) = (-1)^{r(G)} \cdot X^{k(G)} \cdot T(G; 1 - X, 0)$$

- To compute the coefficients of $\chi(G; X)$ from $T(G; X, Y)$ we have to know the parity of $r(G)$ and the number of connected components of $G$.

- For connected graphs $k(G) = 1$ and $r(G) = |V| - 1$. 
Introducing auxiliary parameters $S$

Let $S = \{S_1(G), \ldots, S_t(G)\}$ be graph parameters (polynomials), and $C$ a graph property.

Let $P(G, \bar{X})$ and let $Q(G, \bar{Y})$ be two graph polynomials.

**Definition 7**
We say that $Q$ determines $P$ relative to $S$ over $C$, or $Q$ is at least as distinctive than $P$ relative to $S$ over $C$, and write $P \preceq_{S,C} Q$ if for all graphs $G_1, G_2 \in C$ with $S_i(G_1) = S_i(G_2)$: $i \leq t$ we have

$$Q(G_1) = Q(G_2) \text{ implies that } P(G_1) = Q(P_2).$$

**Definition 8**
We say that $Q$ determines coefficient-wise $P$ relative to $S$ over $(C)$ and write $P \preceq_{S,(C)} Q$ if there is a function $F : (\mathbb{Z}^{<\omega})^{t+1} \to \mathbb{Z}^{<\omega}$ such that for all graphs $G \in \mathcal{P}$

$$F(cS_1(G), \ldots, cS_t(G), cQ(G)) = cP(G)$$

The equivalence relations $P \sim_{r.d.p.} S, Q$ and $P \sim_{relcoeff} S, Q$, are defined as usual.
Theorem 9 \( P \preceq^S_{\text{relcoeff}} Q \iff P \preceq^S_{\text{r.d.p.}} Q \)

The proof is left as an exercise!
Conclusion of Lecture 11

We have established a framework for comparing graph polynomials.

What remains to do?

- In the seminar 238901 next semester
  - Comparing uniform sequences of polynomials.
  - Introducing complexity.

- In Lecture 12
  - Introducing Logic
  - Linear recurrences for graph polynomials