Lecture 7

Translation Schemes: Main definitions and examples

- The framework of translation schemes
  - The induced maps
  - The fundamental lemma
  - Reductions
- The Museum of examples

Definition 1 (Translation Schemes Φ)

- Let τ and $\sigma = \{R_1, \ldots, R_m\}$ be two vocabularies with $\rho(R_i)$ be the arity of R_i.

- Let $\mathcal{L}$ be a fragment of SOL, such as FOL, $MSOL$, $\exists MSOL$, etc.

- Let $\Phi = \langle \phi, \psi_1, \ldots, \psi_m \rangle$ be formulae of $\mathcal{L}(\tau)$ such that ϕ has exactly k distinct free first order variables and each ψ_i has $k\rho(R_i)$ distinct free first order variables.
  We say that Φ is k–**feasible**
  (for σ over τ).

- A k-feasible $\Phi = \langle \phi, \psi_1, \ldots, \psi_m \rangle$ is called a k–τ–σ–$\mathcal{L}$–**translation scheme** or, in short, a **translation scheme**, if the parameters are clear in the context.

Distinctions

If $k = 1$ we speak of **scalar** or **non–vectorized** translation schemes.

If $k \geq 2$ we speak of **vectorized** translation schemes.

If ϕ is such that $\forall \bar{x} \phi(\bar{x})$ is a tautology (always true) the translation scheme is **not relativized** otherwise it is **relativized**.

A translation scheme is **simple** if it is neither relativized nor vectorized.

Example 2 (τ_{words_3} and τ_{graphs})

τ_{words_3} consists of $\{R_<, P_0, P_1, P_2\}$ for three letters $\{0, 1, 2\}$.

τ_{graphs} consists of $\{E\}$

Put $k = 1$,
$\phi_1(x) = (P_0(x) \vee P_1(x))$ and
$\psi_E(x, y) = (P_0(x) \wedge P_1(y))$

$$\Phi_1 = \langle \phi_1(x), \psi_E(x, y) \rangle$$

is a **scalar** and **relativized** translation scheme in FOL.

If instead we look at $\phi_2(x) = (x \approx x)$ then

$$\Phi_2 = \langle \phi_2(x), \psi_E(x, y) \rangle$$

is a **simple** translation scheme.

Example 3 (τ_{words_2} and τ_{grids})

τ_{words_2} consists of $\{R_<, P_0, P_1\}$

τ_{grids} consists of $\{E_{NS}, E_{EW}\}$

Put $k = 2$,
$\phi(x) = ((x \approx x) \wedge (y \approx y))$
$\psi_{E_{NS}}(x_1, x_2, y_1, y_2) = (R_<(x_1, x_2) \wedge y_1 \approx y_2)$
$\psi_{E_{ES}}(x_1, x_2, y_1, y_2) = (R_<(y_1, y_2) \wedge x_1 \approx x_2)$

$$\Phi_3 =$$
$$\langle \phi(x, y), \psi_{E_{NS}}(x_1, x_2, y_1, y_2), \psi_{E_{EW}}(x_1, x_2, y_1, y_2) \rangle$$

is a **vectorized** but **not** relativized translation scheme in FOL.

Definition 4 (The induced transduction $\Phi^\star$)

Given a translation scheme Φ

$$\Phi^\star : Str(\tau) \to Str(\sigma)$$

is a (partial) function from τ–structures to σ–structures defined by $\Phi^\star(\mathcal{A}) = \mathcal{A}_\Phi$ and

1. the universe of $\mathcal{A}_\Phi$ is the set $A_\Phi = \{\bar{a} \in A^k : \mathcal{A} \models \phi(\bar{a})\}$;

2. the interpretation of R_i in $\mathcal{A}_\Phi$ is the set

   $$\mathcal{A}_\Phi(R_i) = \{\bar{a} \in {A_\Phi}^{\rho(R_i)\cdot k} : \mathcal{A} \models \psi_i(\bar{a})\}.$$

   $\mathcal{A}_\Phi$ is a σ–structure of cardinality at most $\mid A \mid^k$.

As Φ is k–feasible for σ over τ, $\Phi^\star(\mathcal{A})$ is defined iff $\mathcal{A} \models \exists \bar{x} \phi$.

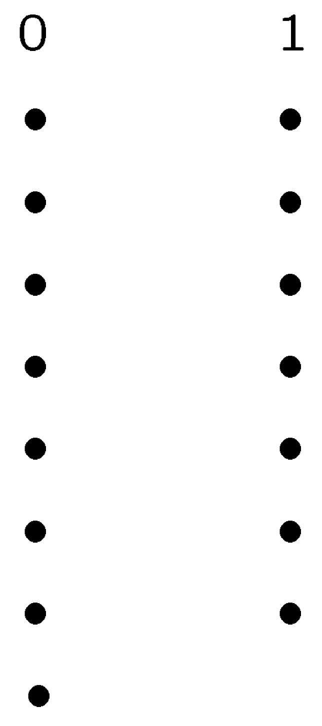
Example 5 (Words and graphs)

Let is compute $\Phi_1^\star$.

For the word

$$100102010200102211 1$$

we get the graph



(1)

Example 6 (Words and grids)

Let is compute $\Phi_3^\star$.

For a word

$$0110101001$$

we get

$$
\begin{array}{c}
\bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \\
\bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \\
\bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \\
\bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \\
\bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \\
\bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \\
\bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \\
\bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \\
\bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \\
\bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet
\end{array}
$$

This is independent of the letters $\P 0, 1\}$.

Definition 7 (The induced translation $\Phi^\sharp$)

Given a translation scheme Φ we define a function $\Phi^\sharp : \mathcal{L}(\sigma) \to \mathcal{L}(\tau)$ from $\mathcal{L}(\sigma)$–formulae to $\mathcal{L}(\tau)$–formulae inductively as follows:

- For $R_i \in \sigma$ and $\theta = R_i(x_1, \ldots, x_m)$ let $x_{j,h}$ be new variables with $i \leq m$ and $h \leq k$ and denote by $\bar{x}_i = \langle x_{i,1}, \ldots, x_{i,k} \rangle$. We put

$$\Phi^\sharp(\theta) = \left(\psi_i(\bar{x}_1, \ldots, \bar{x}_m) \wedge \bigwedge_i \phi(\bar{x}_i) \right)$$

- This also works for equality and relation variables U instead of relation symbols R.

Definition 7 (Continued: booleans)

For the boolean connectives, the translation distributes, i.e.

- if $\theta = (\theta_1 \vee \theta_2)$ then
$$\Phi_\sharp(\theta) = (\Phi_\sharp(\theta_1) \vee \Phi_\sharp(\theta_2))$$
- if $\theta = \neg\theta_1$ then
$$\Phi_\sharp(\theta) = \Phi_\sharp(\neg\theta_1)$$
- similarly for $\wedge$ and $\rightarrow$.

Definition 7 (Continued: quantification)

- For the existential quantifier, we use relativization to ϕ:
  If $\theta = \exists y \theta_1$, let $\bar{y} = \langle y_1, \ldots, y_k \rangle$ be new variables. We put

  $$\theta_\Phi = \exists \bar{y}(\phi(\bar{y}) \wedge (\theta_1)_\Phi).$$

  This concludes the inductive definition for first order logic FOL.

- For second order quantification of variables U of arity ℓ and $\bar{a}$ a vector of length ℓ of first order variables or constants, we translate $U(\bar{a})$ by treating U as a relation symbol above and put

  $$\theta_\Phi = \exists V (\forall \bar{v}(V(\bar{v}) \rightarrow$$
  $$(\phi(\bar{v_1}) \wedge \ldots \phi(\bar{v}_\ell) \wedge (\theta_1)_\Phi)))$$

Example 8 (Computing $\Phi_1^\sharp$)

Recall

$$\Phi_1 = \langle \phi_1(x), \psi_E(x, y) \rangle$$

with $k = 1$,
$\phi_1(x) = (P_0(x) \vee P_1(x))$ and
$\psi_E(x, y) = (P_0(x) \wedge P_1(y))$

Let θ_{conn} be the formula which says the graph is connected:

$$\neg (\exists U\, (\exists x \neg U(x) \wedge \forall x \forall y (U(x) \wedge E(x, y) \rightarrow U(y))))$$

Example 8 (Continued)

- $U(x)$ is replaced by

$$(\phi_1(x) \wedge U(x)) = ((P_0(x) \vee P_1(x)) \wedge U(x))$$

- $E(x, y)$ is replaced by

$$(\phi_1(x) \wedge \phi_1(y) \wedge E(x, y)) =$$
$$((P_0(x) \vee P_1(x)) \wedge (P_0(y) \vee P_1(y)) \wedge E(x, y))$$

- $(x \approx y)$ is replaced by

$$(\phi_1(x) \wedge \phi_1(y) \wedge (x \approx y)) =$$
$$((P_0(x) \vee P_1(x)) \wedge (P_0(y) \vee P_1(y)) \wedge (x \approx y))$$

- Then we proceed inductively.

$(x \approx y)$ does not occur in θ_{conn}.

Proposition 9
(Preservation of tautologies I)

Let $\mathcal{L}$ be First Order Logic FOL.

$$\Phi = \langle \phi, \psi_1, \ldots, \psi_m \rangle$$

be a k–$(\tau - \sigma)$—$\mathcal{L}$–translation scheme, which is not relativizing, i.e. $\forall \bar{x} \phi(\bar{x})$ is a tautology. Let θ a σ-formula.

- If θ is a tautology (not satisfiable), so is $\Phi^\sharp(\theta)$.

- If ϕ is not a tautology, this is not true.

- There are formulas θ which are not tautologies (are satsifiable), such that $\Phi^\sharp(\theta)$ is a tautology (is not satisfiable).

Proof of proposition 9

Proof:
For FOL, the first two parts are by straight induction using the completeness theorem. What we observe is that proof sequences translate properly using $\Phi^\sharp$.

Generalizing to other logics needs regularity conditions.

If ϕ is not a tautology, $\exists x(x = x)$ is a tautology, but $\Phi^\sharp(\exists x(x = x)) = \exists x \phi(x) \wedge x = x$ is not a tautology.

Now let $\Phi = \langle \psi_R, \psi_S \rangle$ be defined by

$$\psi_R(x) = P(x) \text{ and } \psi_S(x) = \neg P(x).$$

$\exists x \theta_1$ be $R(x) \wedge S(x)$ and $\exists x \theta_2$ be $R(x) \vee S(x)$ are both satisfiable but not tautolgies. But $\Phi^\sharp(\theta_1)$ is not satisfiable and $\Phi^\sharp(\theta_2)$ is a tautology. $Q.E.D.$

Theorem 10 (Fundamental Property)

Let $\Phi = \langle \phi, \psi_1, \ldots, \psi_m \rangle$ be a k–(τ–σ)–translation scheme in a logic $\mathcal{L}$. Then the transduction $\Phi^\star$ and the translation $\Phi^\sharp$ are in linked in $\mathcal{L}$.

In other words, given

- $\mathcal{A}$ be a τ-structure and
- θ be a $\mathcal{L}(\sigma)$–formula.

Then

$$\boxed{\mathcal{A} \models \Phi^\sharp(\theta) \text{ iff } \Phi^\star(\mathcal{A}) \models \theta}$$

Translation Scheme and its induced maps

in the Fundamental Property of theorem 10

Translation scheme Φ		
	$\Phi^\star$	
τ-structure	$\longrightarrow$	σ-structure
$\mathcal{A}$		$\Phi^\star(\mathcal{A})$
τ-formulae	$\longleftarrow$	σ-formulae
	$\Phi^\sharp$	
$\Phi^\sharp(\theta)$		θ
$\mathcal{A} \models \Phi^\sharp(\theta)$	iff	$\Phi^\star(\mathcal{A}) \models \theta$

Definition 11 ($\mathcal{L}$–Reductions)

Let $\mathcal{L}$ be a regular logic and Φ be a $(\tau_1 - \tau_2)$ translation scheme. We are given

- two classes K_1, K_2 of $\tau_1(\tau_2)$-structures closed under isomorphism

We say

1. $\Phi^\star$ is a *weak reduction* of K_1 to K_2 if for every τ_1-structure $\mathfrak{A}$ with $\mathfrak{A} \in K_1$ we have $\Phi^\star(\mathfrak{A}) \in K_2$.

2. $\Phi^\star$ is a *reduction* of K_1 to K_2 if for every τ_1-structure $\mathfrak{A}$, $\mathfrak{A} \in K_1$ iff $\Phi^\star(\mathfrak{A}) \in K_2$.

Definition 11(Continued)

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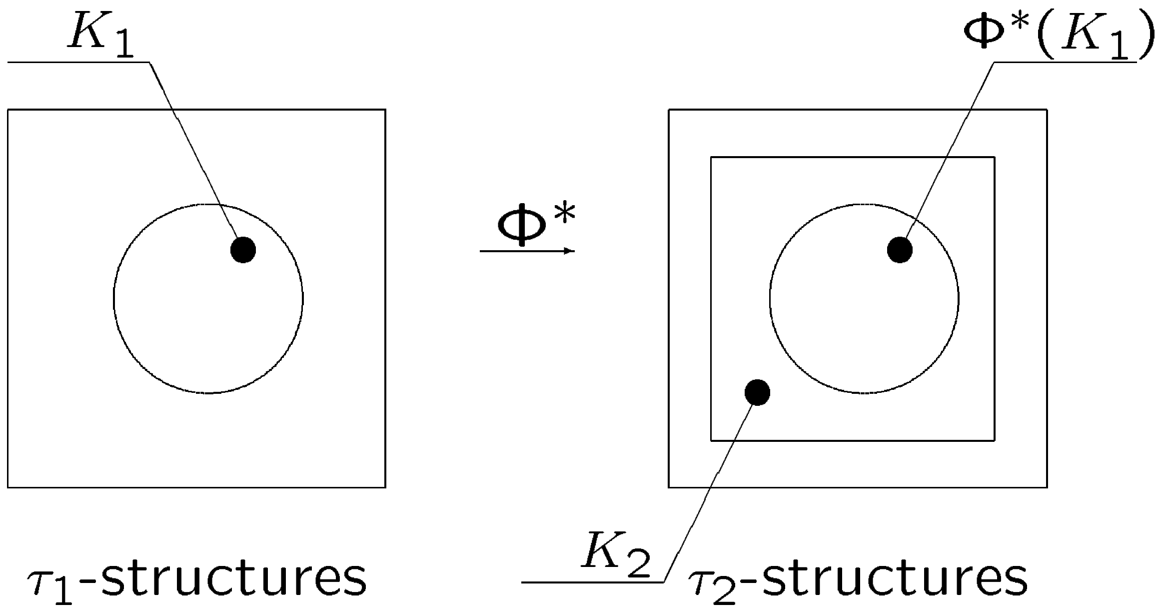
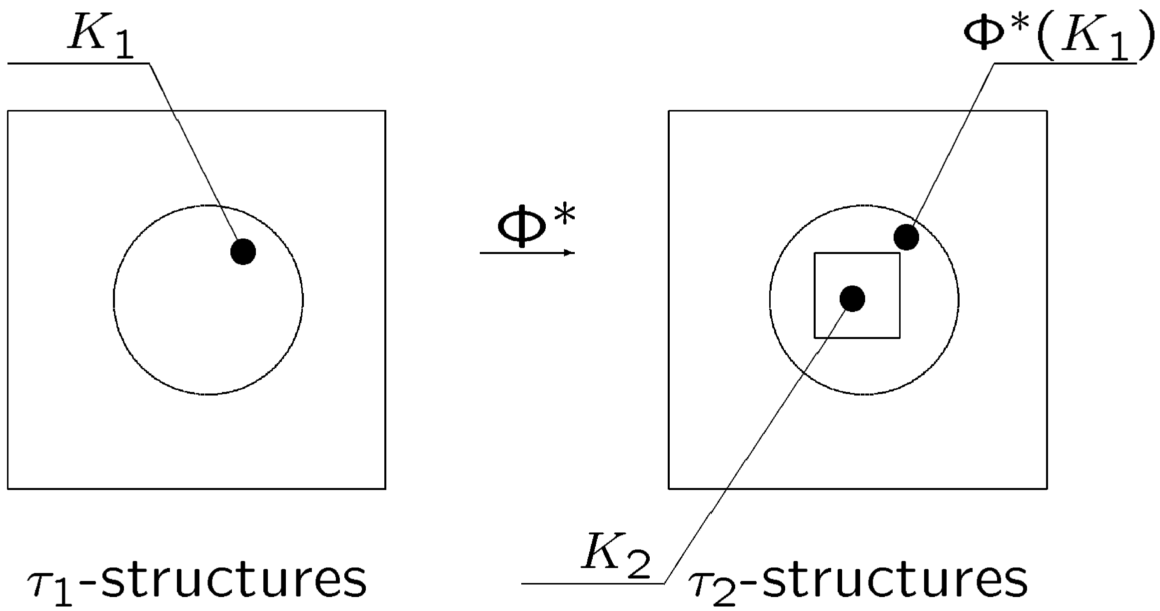
3. $\Phi^\star$ of K_1 to K_2 is *onto* if (additionally) for every $\mathfrak{B} \in K_2$ there is an $\mathfrak{A} \in K_1$ with $\Phi^\star(\mathfrak{A})$ isomorphic to $\mathfrak{B}$.

4. By abuse of language we say $\Phi^\star$ is a *translation of* K_1 *onto* K_2 also if $\Phi^\star$ is not a weak reduction but only $K_2 \subseteq \Phi^\star(K_1)$.

5. We say that Φ induces a reduction (a weak reduction) of K_1 to K_2, if $\Phi^\star$ is a reduction (a weak reduction) of K_1 to K_2. For simplicity, we also say Φ is a reduction (a weak reduction) instead of saying that Φ induces a reduction (a weak reduction).



Weak reduction



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Definition 12 ($\mathcal{L}$–Reducibility)

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1. Let $k \in \mathbf{N}$.
   We say that K_1 is $\mathcal{L}$-k-*reducible* to K_2 ($K_1 \lhd_{\mathcal{L}-k} K_2$), if there is a $\mathcal{L}$-k-translation scheme Φ for τ_2 over τ_1, such that $\Phi^\star$ is a reduction of K_1 to K_2.

2. We say that K_1 is $\mathcal{L}$-*reducible* to K_2 ($K_1 \lhd_{\mathcal{L}} K_2$), if $K_1 \lhd_{\mathcal{L}-k} K_2$ for some $k \in \mathbf{N}$.

3. We say that K_1 is $\mathcal{L}$-*bi-reducible* to K_2 and write $K_1 \bowtie_{\mathcal{L}} K_2$, if $K_1 \lhd_{\mathcal{L}-k} K_2$ and $K_2 \lhd_{\mathcal{L}-k} K_1$ for some $k \in \mathbf{N}$.
   Clearly, bi-reducibility is a symmetric relation.

Theorem 13 (Definability and Reducibility)

Let $\Phi^\star$ be an $\mathcal{L}$-reduction of K_1 to K_2.
If K_2 is $\mathcal{L}$-definable then K_1 is -definable.

Recall that a class of τ-structures K_2 is $\mathcal{L}$-definable if there is a $\mathcal{L}(\tau)$-sentence θ such that $K_2 = Mod(\theta)$.

Proof:

We use the Fundamental Property of Φ.

If K_2 is defined by θ, so K_1 is defined by $\Phi^\sharp(\theta)$.

Proposition 14

Hamiltonian graphs are not $MSOL$-definable (both in τ_{graphs_1} and τ_{graphs_2}).

Proof:

We use Φ_2 from example 2.

$\Phi_2^\star$ is a reduction from words $0^n 1^m$ over $\{0,1\}$ to complete bipartite graphs $K_{n,m}$, which are $MSOL$-defined by θ_{co-bi}.

$K_{n,m}$ is Hamiltonian iff $n = m$.

So, if θ_{hamil} defined all Hamiltonian graphs,

$$\Phi_2^\sharp(\theta_{hamil} \wedge \theta_{co-bi})$$

defined the language $\{0^n 1^n\}$.

But $\{0^n 1^n\}$ is not regular, and hence, by Büchi's theorem, not $MSOL$-definable.

Q.E.D.

Proposition 15

Eulerian graphs are not $MSOL$-definable (both in τ_{graphs_1} and τ_{graphs_2}).

Proof: Let SET be the class of finite sets and $ODD \subseteq SET$ those of odd cardinality.
Let $CLIQUE$ be the class of complete graphs.
$CLIQUE$ is FOL-definable by some θ_{clique}.

Let the simple FOL translation scheme Φ be given by
$\phi(x) = (x \approx x)$ and $\psi_E(x, y) = (\neg x \approx y)$.

$\Phi^{\star}$ is a reduction from SET to $CLIQUE$.

Now assume that there is $\theta_{euler} \in MSOL$, with $EULER = Mod(\theta_{euler})$.
Put $\theta = (\theta_{clique} \wedge \theta_{euler})$.
$\Phi^{\sharp}(\theta)$ is equivalent to $\theta_{odd} \in MSOL$.

But this contradicts the fact that ODD ($EVEB$) is not $MSOL$-definable.

Q.E.D.

Proof of theorem 10

We use induction over the construction of θ.

- If all the formulas ϕ, ψ_i of Φ and θ are atomic,
both $\Phi^\star(\mathfrak{A}) = \mathfrak{A}$ and
$\Phi^\sharp(\theta) = \theta$.

- Next we keep θ atomic and assume

$$\Phi = \langle \phi(\bar{x}), \psi_{S_1}(\bar{x}), \ldots \psi_{S_m}(\bar{x}) \rangle$$

$\Phi^\star(\mathfrak{A}) \models S_i(\bar{a})$ iff $\mathfrak{A} \models \psi_{S_i}(\bar{a})$
by definition of $\Phi^\star$.

- Now the induction on θ uses that $\Phi^\sharp$ commutes with the logical constructs.

Q.E.D.

Proposition 16
(Preservation of tautologies II)

Let $\mathcal{L}$ be First Order Logic FOL.

$$\Phi = \langle \phi, \psi_1, \ldots, \psi_m \rangle$$

be a k–$(\tau - \sigma)$—$\mathcal{L}$–translation scheme. Let θ a σ-formula.

Assume that $\Phi^\star$ is onto all σ-structures, i.e. for every σ-structure $\mathfrak{B}$ there is a τ-structure $\mathfrak{A}$ such that $\Phi^\star(\mathfrak{A}) = cong \mathfrak{B}$

- If θ is a tautology, so is $\Phi^\sharp(\theta)$.

- If additionally $\exists \bar{x} \phi(\bar{x})$ is a tautology and $\Phi\sharp(\theta)$ is a tautology then θ is a tautology.

Proof:
Use the fundamental property. $Q.E.D.$
Note that here the proof is semantical.

Example 17 (Renaming)

One of the simplest translations encountered in logic is the renaming of basic relations.

Let $\tau_1 = \{R_i : i \leq k\}$ and $\tau_2 = \{S_i : i \leq k\}$, where R_i and S_i are of the same arity, respectively.

Let Φ be the (τ_1, τ_2) translation scheme given by $\Phi = \langle x = x, R_1(\bar{u}), \ldots, R_k(\bar{v})\rangle$.

Such a translation scheme and as well as its induced maps $\Phi^\star$ and $\Phi^\sharp$ are called **renaming**.

Example 18 (Cartesian Product)

Let us consider one example of vectorized translation scheme that defines Cartesian Product.

For simplicity, we assume that $k = 2$.

Let $\tau_1 = \{R_1(x_1, x_2)\}$ with R_1 binary and $\tau_2 = \{R_2(x_1, x_2)\}$ with R_2 binary.

$$\Phi = \langle (x_1 = x_1 \lor x_2 = x_2), \\ (R_1(x_1, x_2) \land R_2(x_3, x_4)) \rangle$$

It is easy to see that $\Phi^{\star}(\mathcal{A})$ is isomorphic to the Cartesian product $\mathcal{A}^2$.

The n-hold Cartesian product is defined in the same way.

Example 19 (Graphs)

$Graphs_1$ is the class of structures of the form $\langle V, E\rangle$ where E is a binary irreflexiv relation on the set of vertices V.

$Graphs_2$ is the class of structures of the form $\langle V \sqcup E; Src(v,e), Tgt(v,e)\rangle$ with the universe consisting of **disjoint** sets of vertices and edges and $Src(v,e)$ ($Tgt(v,e)$) indicates that v is the source (target) of the directed edge e.

For a graph G we denote its representations by G_i for $G_i \in Graphs_i$ respectively.

We define a scalar translation scheme $\Phi = \langle \phi, \psi_E\rangle$ from $Graphs_2$ to $Graphs_1$ by

$$\phi(v) = (\exists e(Src(v,e) \vee eTgt(v,e)) \vee$$
$$(v = v \wedge \neg\exists x(Src(x,v) \vee Tgt(x,v))$$
$$\phi_E(x,y) = \exists e\,((Src(x,e) \wedge Tgt(y,e))$$

Clearly, for every graph G we have

$$\Phi^\star(G_2) \cong G_1$$

Theorem 20 (Complexity of transductions)

If Φ is in FOL (or $\exists HornSOL$)
then $\Phi^\star$ is computable in polynomial time.

Proof:

We test all k-tuples $\bar{a}$ in $\mathfrak{A}$ of size n for

$$\mathfrak{A} \models \phi(\bar{a})$$

This takes $n^k \cdot TIME(\mathfrak{A}, \phi)$ time.

But we know that $TIME(\mathfrak{A}, \phi)$ is a polynomial in n.

For the ψ_{S_i} this is the same.

Q.E.D.

By a theorem of Grädel, this also holds for $HornSOL$, cf. the project page.