THREE TREE-PATHS

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Abstract

Itai and Rodeh [IR] have proved that for any 2-connected graph G and any vertex $s \in G$ there are two spanning trees such that the paths from any other vertex to s on the trees are disjoint. In this paper the result is generalized to 3-connected graphs.

1. Introduction.

A *graph* $G=(V(G),E(G))$ consists of a set of vertices $V(G)$ and a set $E(G)$ of unordered pairs of $V(G)$ called *edges.* An $x-y$ *path* $\pi[x,y]$ in a graph G is a sequence of distinct vertices of $V(G)$ $(x = v_0,v_1,...,v_m = y)$ such that $(v_{i-1},v_i) \in E(G)$ $(i=1,...,m)$. Two paths $\pi_1[x_1,y_1]$ and $\pi_2[x_2,y_2]$ are *disjoint* if they have no common vertices except perhaps $\{x_1,y_1\} \cap \{x_2,y_2\}$. Let $\kappa(x,y;G)$ denote the maximum number of disjoint $x-y$ paths in G. A graph G is *k-connected* if between any two vertices of G there are at least k disjoint paths. The connectivity, $\kappa(G)$, of G is the maximum integer k such that G is k-connected, i.e., $\min\{\kappa(x,y;G) : x,y \in V(G)\} = k$. If $v \in V(G)$ we denote by $\Gamma(v)$ the set of all vertices adjacent to v while $\Gamma_E(v)$ denotes the set of edges adjacent to v. The degree of v, $deg(v;G)$, is equal to the cardinality of $\Gamma_E(v)$. A vertex v is a *leaf* if $deg(v;G)=1$.

We abuse the notation and say that an edge or a vertex belongs to a graph ($e \in G$ or $v \in G$ instead of $e \in E(G)$ or $v \in V(G)$). Also, $G-e$ denotes the graph from which e was deleted from the edge set (likewise $G+e$). For all other notations refer to Bondy & Murty ([BM76]).

Let T be a spanning tree of G and $v,w \in V(G)$ then $T[v,w]$ is the path from v to w on T. The following result appears in [IR]:

Theorem 1.1

If G is a 2-connected graph and $s \in V(G)$, then there exist two spanning trees of G, T_1, T_2, such that for every vertex $v \in G$ the paths $T_1[v,s]$ and $T_2[v,s]$ are disjoint.

Itai and Rodeh [IR] used this result to develop distributed algorithms which are resilient to the failure of a single line (or processor). To increase reliability more spanning trees are needed. The purpose of this paper is to prove the following:

Theorem 1.2

If G is a 3-connected graph and $s \in V(G)$, then there exist three spanning trees of G, T_i $(i=1,2,3)$, such that for every vertex $v \in G$, the paths $T_i[v,s]$ $(i=1,2,3)$ are disjoint.

We start the paper by proving general properties of 3-connected graphs (Section 2). To prove the theorem, in Section 3 we define a property, *Extended 3-Tree Path property (E3TP)*, which implies the existence of the three trees. The remainder of the paper shows that 3-connectivity implies E3TP. To that end, we investigate the structure of a *minimal counterexample (MCE)*. In Section 4 we show that in every MCE, $deg(s)$=3. In Section 5 it is shown that an MCE cannot contain certain subgraphs. The proof is completed in Section 6. While Section 7 contains the conclusions.

Our proof methodology is to obtain from an MCE, G, a smaller graph G', then show that G' is 3-connected, thus by definition of MCE, G' satisfies E3TP. Thus G' contains three trees, which we modify to get three trees in G.

2. Connectivity

Let $U \subseteq V(G)$ then $E(U;G)$ consists of all the edges of G both endpoints of which belong to U and the *subgraph induced by* U is the graph $G[U]=(U, E(U;G))$. A set $S \subseteq V(G)$ is a *separation set* of G if its deletion from G increases the number of connected components. A separation set is a *minimum separation set* if its cardinality is minimum.

The *concatenation* $\pi\sigma$ of two edge-disjoint paths $\pi=\pi[v,x]$ = $(v=v_0,...,v_n=x)$ and $\sigma=\sigma[x,w]$ = $(x=v_n, v_{n+1},...,v_m)$ is the $v-w$ path obtained by deleting cycles from the sequence $(v=v_0,...,v_m=w)$.

Lemma 2.1

Let G be a 3-connected graph, $S=\{v_1,v_2,v_3\}$ a separation set of G and $e=(v_1,v_2)$. Then for all vertices $x,y \in V(G)-\{v_1,v_2\}$, $\kappa(x,y;G-e)$ = 3.

Proof

Case 1: $v_3\{x,y\}$: Thus, $x,y \in V(G)-S$. If x and y belong to different connected components of $G-S$ then there exist three $x-y$ paths which do not use e. Thus, assume x and y are in the same component. At most one $x-y$ path uses e. In this path, e can be replaced by a v_1-v_2 path which except for its endpoints lies entirely in a component not containing x and y.

Case 2: $v_3 \in \{x,y\}$: W.l.g., $v_3{=}y$. In G there are three $x{-}v_3$ paths. At most one of them, say π_1, uses e. (If none uses e, we are done.) The other two paths lie entirely in the connected component of $G{-}S$ which contains x. Let $\pi_1 = \pi_1[x,v_i](v_i,v_{3-i})\pi_1[v_{3-i},v_3]$, for $i{=}1$ or $i{=}2$. Let σ be a $v_i{-}v_3$ path in the connected component of $G{-}S$ not containing x. Then replace π_1 by the path $\pi_1'{=}\pi_1[x,v_i]\sigma$ to obtain three $x{-}v_3$ paths in $G{-}e$. $\square$

The following two lemmas follow from Menger's Theorem,

Lemma 2.2:

Let $v \in V(G)$ and $deg(v;G){\ge}k$. If for every two vertices $x,y{\ne}v$, $\kappa(x,y;G){\ge}k$ then $\kappa(G){\ge}k$.

Proof

If $\kappa(G){<}k$ there exists a set W of cardinality $\le k{-}1$ separating v from some other vertex, say x. Since $deg(v){\ge}k$, v has a neighbor $y\,W$. Thus W separates y from x, which implies $\kappa(x,y;G){\le}k{-}1$. $\square$

An edge e is *subdivided* when it is deleted and replaced by a path of length two connecting its endpoints, the internal vertex of this path being a new vertex. The graph G is a *subdivision* of H if G can be obtained from H by a series of edge subdivisions. Let $h(G)$ be the minimum size graph H such that G is a subdivision of H.

Lemma 2.3

Let G be a 3-connected graph, $S{=}\{v_1,v_2,v_3\}$ a separating set and $e{=}(v_1,v_2)$. Then $H{=}h(G{-}e)$ is 3-connected.

Proof

It is sufficient to show that every two vertices $x,y \in V(H)$ are connected by three disjoint paths. By Lemma 2.1, $\{x,y\}\cap\{v_1,v_2\}{\ne}\emptyset$. Thus, w.l.g., $x = v_1$,

Case 1: $deg(v_1;G){=}3$: we are done since $deg(v_1;G{-}e){=}2$, implying that $v_1 H$.

Case 2: $deg(v_1;G){>}3$: If $deg(y;G{-}e){=}2$ then $y{=}v_2$ and by reversing the roles of x and y we return to Case 1. Thus, $deg(y;G{-}e) \ge 3$. Let $W{=}\{w_1,w_2\}{\subseteq}V(H)$ separate between v_1 and y in $G{-}e$ (thus v_1

and y are not connected in $G-e$). Since $deg(v_1;G-e)\geq 3$, in $G-e$ v_1 has a neighbor $u_1 W \cup \{y\}$. If $y \neq v_2$, W separates between u_1 and y contrary to Lemma 2.1. Otherwise, $y = v_2$ has a neighbor $u_2 W \cup \{v_1\}$ and W separates between u_1 and u_2 again contradicting Lemma 2.1. $\square$

Remark: Figure 2.1 shows that the condition $e \in E(S)$ is necessary.

Let $e = (x,y)$ be an edge of G. The *contraction of* e *in* G (G/e) is the graph resulting from G by replacing the vertices x,y by a single vertex $\overline{xy}$ and connecting it to $\Gamma(x;G) \cup \Gamma(y;G) - \{x,y\}$.

Corollary 2.4:

Let G be a 3-connected graph and $e = (v_1,v_2) \in E(G)$. Then either G/e or $h(G-e)$ is 3-connected.

Proof

If G/e is only 2-connected then v_1 and v_2 belong to some minimum separating set $S = \{v_1,v_2,v_3\}$, to which the previous lemma may be applied. $\square$

We conclude this section with a lemma to be used in Section 6.

Let $x \in G$ and $S \subseteq V(G)$. An $x-S$ *fan* is a set of disjoint paths $\pi_1[x,y_1],...,\pi_k[x,y_k]$ were all the vertices y_i are distinct vertices of S; k is the *size* of the fan.

Lemma 2.5

Let $B \subseteq V(G)$ be a set of vertices such that $\kappa(x,y;G) = k$ for all $x,y \in B$. If for every other vertex v there is a $v-B$ fan of size k, then $\kappa(G) = k$.

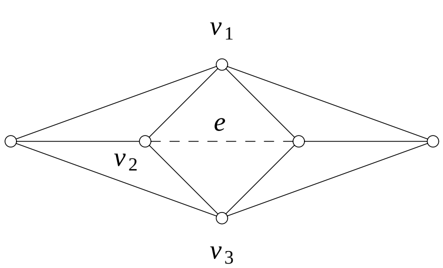


Figure 2.1

Proof

Let v,w be two vertices of G and S a separation set of G which separates v from w, such that $|S|<k$. By our assumption S cannot separate all $v-B$ paths nor all $w-B$ paths. Thus, for some $x,y \in B$ there are two paths $\pi[v,x]$ and $\sigma[w,y]$ which do not pass through vertices of S. As $x,y \in B$ there are k disjoint $x-y$ paths, and S cannot separate all of them. Let $\tau[x,y]$ be an $x-y$ path not passing through S. We conclude that $\rho = \pi[v,x]\tau[x,y]\sigma[y,w]$ is a $v-w$ path not passing through S (ρ is a path since by our definition of concatenation of paths cycles are deleted); thus S does not separate v from w. $\square$

3. Extended three tree path property.

Let S be a set. The relation $\prec \subseteq S \times S$ is an *order on S* if $\prec$ is irreflexive, antisymmetric, transitive and total, i.e., for all $u,v \in S$ either $v \prec u$ or $u \prec v$.

To prove Theorem 1.2 we need the following definition of an $s-t$ ordering which is equivalent to the $s-t$ numbering found in [Ev] and used in [IR]. Let $e=(s,t)$ be an edge in a graph G. An *$s-t$ ordering* of G is an order $\prec$ on the vertices of G such that:

(S1) s is minimum and t is maximum, i.e. for all $v \neq s,t$ $\quad s \prec v \prec t$.

(S2) For every vertex $v \in V(G) - \{s,t\}$ there are two vertices $v_1,v_2 \in \Gamma(v)$ such that $v_1 \prec v \prec v_2$.

In [Ev] it is proven that such a ordering exists for all 2-connected graphs.

Let $\prec$ be an order on the vertices of the graph G. A path $\pi=(v_1, \ldots, v_n)$ *$\prec$–increases* (*$\prec$–decreases*) if $v_1 \prec v_2 \prec \cdots \prec v_n$ $\quad (v_n \prec v_{n-1} \prec \cdots \prec v_1)$. Let s be the minimum vertex of $\prec$. A pair of spanning trees, (T_1,T_2) *are ordered by* $\prec$ if for every vertex v the path $T_1[v,s]$ $\prec$-decreases and the path $T_2[v,s]-s$ $\prec$-increases. (In this case $T_2[v,s]-s = T_2[v,t]$.) The motivation for this definition is that when $u \prec v$, $T_1[u,s]$ and $T_2[v,s]$ are disjoint. The following lemma is useful in proving that an order is an $s-t$ ordering.

Lemma 3.1

If the pair of spanning trees (T_1, T_2) is ordered by $\prec$ which satisfies (S1) for some edge (s,t) then $\prec$ is an $s-t$ ordering.

Proof

Let $v \neq s,t$ be a vertex of G. Since (T_1,T_2) are ordered by $\prec$, the path $T_1[v,s] = (v,v_1)T_1[v_1,s]$ is $\prec$-decreasing. Therefore v_1 is a neighbor of v satisfying $v_1 \prec v$. Considering $T_2[v,t]$ provides us with another neighbor v_2 satisfying $v \prec v_2$. $\square$

We now show a simple application of this lemma to be used in the proof of our last theorem.

Lemma 3.2

Let T_1, T_2 be spanning trees of a graph G and $\prec$ an $s-t$ order of G. If (T_1,T_2) is ordered by $\prec$ and $\Gamma(s;T_1)=\{v\}$ then there exists an $s-v$ order $\prec'$ such that (T_2,T_1) is ordered by $\prec'$.

Proof

To define $\prec'$, let s be the $\prec'$-minimum and v the $\prec'$-maximum. For all other vertices u,w if $u \prec w$ then define $w \prec' u$. Since a $\prec$-increasing (decreasing) path is $\prec'$-decreasing (increasing), (T_2,T_1) is ordered by $\prec'$, so by the previous lemma $\prec'$ is an $s-v$ ordering. $\square$

Let $s \in V(G)$, and $e_1,e_2 \in \Gamma_E(s)$. The quadruple (G,s,e_1,e_2) satisfies the *Extended 3-Tree Path* property $(E3TP)$ if there are three trees (T_1,T_2,T_3) such that:

(E1) T_i are spanning trees of G $(i=1,2,3)$.

(E2) For all $v \in G$ $T_3[v,s]$ is disjoint of $T_1[v,s]$ and $T_2[v,s]$.

(E3) $E(T_i) \cap \Gamma_E(s) = \{e_i\}$ for $i = 1,2$.

(E4) If $e_2 = (s,t)$ then there exists an $s-t$ ordering $\prec$ of G such that (T_1,T_2) is ordered by $\prec$.

Note that by (E4) $T_1[v,s]$ and $T_2[v,s]$ are disjoint, and in fact $\prec$ was introduced to facilitate proving that.

Lemma 3.3

Let G be a graph. If for every $s \in V(G)$ there exist $e_1, e_2 \in \Gamma_E(s)$, such that (G, s, e_1, e_2) satisfies E3TP then G is 3-connected.

Proof

We show that between every $v, w \in V$ there exist three disjoint paths. Let $e_1, e_2 \in \Gamma_E(v)$. Since (G, v, e_1, e_2) satisfies E3TP construct the three trees. The desired paths are $T_i[w, v]$ $(i=1,2,3)$. $\square$
Note that some 2-connected graphs G may satisfy E3TP for some s, e_1 and e_2 (Figure 3.1).

Our goal will be to prove the converse of this lemma, namely, for every 3-connected graph G, $s \in V(G)$ and $e_1, e_2 \in \Gamma_E(s)$, (G, s, e_1, e_2) satisfies E3TP. To this end, (G, s, e_1, e_2) is a *counterexample* if G is a 3-connected graph, $s \in V(G)$, $e_1, e_2 \in \Gamma_E(s)$ and (G, s, e_1, e_2) does not satisfy E3TP. A *minimum-counter-example* (MCE) is a counterexample for which $|V(G)| + |E(G)|$ is minimum. A graph G is an MCE if there exist s, e_1 and e_2 s.t. (G, s, e_1, e_2) is an MCE. To satisfy our goal of proving Theorem 1.2, we show that there exists no MCE.

4. Reducing the degree of the root of an MCE

4.1 Edge contraction in MCE.

In this and the following section we investigate the structure of an MCE. Here we consider contracting an edge incident with s, and in subsection 4.2 we delete such an edge.

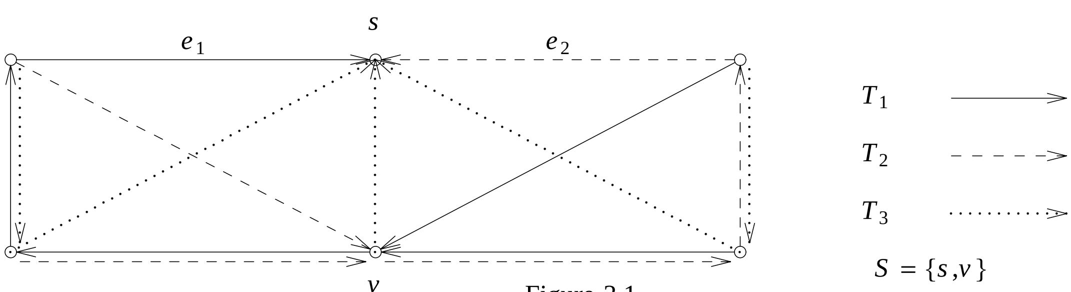


Figure 3.1

Lemma 4.1:

Let $e = (s, p) \neq e_1, e_2$. If $\kappa(G/e) = 3$ then (G, s, e_1, e_2) is not an MCE.

Proof

For the sake of contradiction assume that (G, s, e_1, e_2) is an MCE. By hypothesis G/e is 3-connected. If $e_i=(s,v_i)$ then let $e_i'=(\overline{sp},v_i)$. By the minimality of G, $(G/e, \overline{sp}, e_1', e_2')$ satisfies E3TP. Let T_i' $(i=1,2,3)$ be the corresponding trees in G/e and $\prec'$ the $\overline{sp}-t$ ordering. As $deg(p;G) \geq 3$, p has two neighbors $p_1, p_2 \neq s$. Assume, w.l.g., that $p_1 \prec' p_2$.

Define now $T_i = T_i' - e_i' + e_i + (p, p_i)$ $(i=1,2)$. Note that $T_i[p,s] = (p,p_i)T_i[p_i,s]$. The tree T_3 is obtained from T_3' by adding the edge (s,p) and replacing each edge of the form $(v,\overline{sp})$ by the corresponding edge (either (v,s) or (v,p)) of G.

To define the $s-t$ ordering in G let $\prec$ be any extension of $\prec'$ to all of $V(G)$ such that $p_1 \prec p \prec p_2$.

We prove now that the new trees satisfy E3TP in G.

(E1) The subgraph T_i is a spanning tree of G since $|V(T_i)| = |V(G/e)|+1 = |V(G)|$ and $|E(T_i)| = |E(T_i')| + 1 = (|V(G/e)|-1)+1 = |V(G)|-1$.

(E2) We show that $T_1[v,s]$ and $T_3[v,s]$ are disjoint, the proof for $T_2[v,s]$ is identical. For $v \neq p$: $T_1[v,s] = T_1'[v,\overline{sp}] - e_1' + e_1$. Thus, since $T_1'[v,\overline{sp}]$ and $T_3'[v,\overline{sp}]$ are disjoint, $T_1[v,s] \cap T_3[v,s] \subseteq \{v,s,p\}$. Since p is a leaf of T_1 and $v \neq p$, $p \notin V(T_1[v,s])$ and so $T_1[v,s]$ and $T_3[v,s]$ are disjoint.

As for p, by the construction $T_1[p,s] = (p,p_1)T_1'[p_1,\overline{sp}]$, and $T_3[p,s] = (p,s)$. Therefore, also the path $T_1[p,s]$ is disjoint of $T_3[p,s]$.

(E3) Actually no edges were removed from T_i' $i=1,2$ and the only added edge is (p,p_i) which is not adjacent to s.

(E4) By definition $\prec$ is a total order and as $p_1 \prec p \prec p_2$ (S1) holds. By Lemma 3.1 it remains to show that (T_1, T_2) are ordered by $\prec$. For $v \neq p$, $T_i'[v,\overline{sp}] = T_i[v,s]$ $(i=1,2)$. Since $T_1'[v,\overline{sp}]$ is $\prec$-

decreasing then so is $T_1[v,s]$. Thus, it remains to check only for p. $T_1[p,s] = (p,p_1)T_1'[p_1,\overrightarrow{sp}]$ and as $p_1 \prec p$ and $T_1'[p_1,\overrightarrow{sp}]$ is $\prec$-decreasing so is $T_1[p,s]$.

The proof that $T_2[p,t]$ is $\prec$-increasing is identical. □

4.2 Edge removal.

Let $e = (s,p) \neq e_1,e_2$. If $G-e$ is 3-connected then obviously (G,s,e_1,e_2) is not an MCE. In this section we generalize this observation to the case where $deg(p;G-e) = 2$.

Lemma 4.2

Let $s \in G$ with $deg(s;G)>3$ and $e \in \Gamma_E - \{e_1,e_2\}$. If $H=h(G-e)$ is 3-connected then (G,s,e_1,e_2) is not an MCE.

Proof

Assume that (G,s,e_1,e_2) is an MCE. Let $e = (s,p)$. If $deg(p;G)>3$ then $H=G-e$. Since by the hypothesis H is 3-connected, (H,s,e_1,e_2) is a smaller counterexample.

Therefore, $deg(p;G) = 3$. Let $\Gamma(p) = \{p_1,p_2,s\}$. If $p \in V(H)$ then since $deg(p;H)=2$, H is not 3-connected. (This happens only if $(p_1,p_2) \in E(G)$.) Thus $|V(H)| < |V(G)|$ and since (G,s,e_1,e_2) is an MCE, (H,s,e_1,e_2) satisfies E3TP. Let T_i' $(i=1,2,3)$ be the corresponding trees, and $\prec'$ the $s-t$ ordering. Assume, w.l.g., that $p_1 \prec' p_2$. If the edge (p_1,p_2) belongs to T_3' then assume, w.l.g., that $T_3'[p_1,s] = (p_1,p_2)T_3'[p_2,s]$ and define $T_3 = T_3'-(p_1,p_2)+(p_1,p)+(p,s)$. Otherwise, $T_3 = T_3'+(p,s)$. (Thus we may have shortened some paths of T_3.) Now we construct the other two trees: If $(p_2,p_1) \in E(T_1')$ then $T_1 = T_1'-(p_2,p_1)+(p_2,p)+(p,p_1)$. Otherwise, $T_1 = T_1'+(p,p_1)$. If $(p_1,p_2) \in E(T_2')$ then $T_2 = T_2'-(p_1,p_2)+(p_1,p)+(p,p_2)$. Otherwise, $T_2 = T_2'+(p,p_2)$.

To define the $s-t$ ordering in G let $\prec$ be any extension of $\prec'$ to all vertices of G such that $p_1 \prec p \prec p_2$. We prove that the trees T_i $(i=1,2,3)$ satisfy E3TP in G.

(E1) and (E2)

By our construction, replacing the edge (p_1,p_2) by the path (p_1,p,p_2) leaves the trees connected without introducing any cycles. If $(p_1,p_2) \in T_3'[v,s]$ then $(p_1,p_2)T_i[v,s]$ $(i=1,2)$,

$T_3[v,s] \subseteq T_3'[v,s]+(p_1,p)+(p,s)$ and thus is disjoint of $T_i[v,s]$. Otherwise, $T_3[v,s] = T_3'[v,s]$ and therefore, (E1) and (E2) hold for T_j $(j=1,2,3)$. Thus, it remains to prove these conditions only for p. But $T_1[p,s] = (p,p_1)T_1[p_1,s]$, $T_2[p,s] = (p,p_2)T_2[p_2,s]$ and $T_3[p,s] = (p,s)$ are disjoint.

(E3) No edges adjacent to s were added or removed from T_1' and T_2'. Thus this condition remains unchanged.

(E4) First we prove that $\prec$ is an $s-t$ ordering. Since $\prec$ satisfies condition (S1), by Lemma 3.1 it remains to show that (T_1,T_2) are ordered by $\prec$. For p, $T_1[p,s] = (p,p_1)T_1'[p_1,s]$ $p_1 \prec p$ and the path $T_1'[p_1,s]$ $\prec$-decreases. Similarly, since $p \prec p_2$, the path $T_2[p,t] = (p,p_2)T_2'[p_2,t]$ $\prec$-increases. For $v \neq p$ then we have to check only the case where $(p_2,p_1) \in T'_1[v,s]$ (and similarly for $T'_2[v,t]$). As $p_1 \prec' p_2$, p_1 appeared in $T_1'[v,s]$ after p_2. To get $T_1[v,s]$ we replaced the edge (p_2,p_1) by the path (p_2,p,p_1), but $p_1 \prec p \prec p_2$, thus the order on $T_1[v,s]$ is preserved. $\square$

4.3 The root-degree of an MCE.

In this section we prove the following,

Lemma 4.3

If (G,s,e_1,e_2) is an MCE then $deg(s) = 3$.

Proof

Let $e \in \Gamma_E(s)-\{e_1,e_2\}$. By Corollary 2.4 either G/e or $h(G-e)$ is 3-connected. If G/e is 3-connected then by Lemma 4.1 (G,s,e_1,e_2) is not an MCE. Thus, $\kappa(h(G-e))\geq 3$. If $deg(s)>3$ then by Lemma 4.2 (G,s,e_1,e_2) is not an MCE. $\square$

In the remaining sections we assume $deg(s) = 3$.

Lemma 4.4

Let $\Gamma_E(s) = \{s_1,s_2,s_3\}$ and $(G,s,(s,s_1),(s,s_2))$ satisfy E3TP. Let T_1,T_2,T_3 be the corresponding trees. Then for $1\leq i,j \leq 3$ and $i \neq j$, s_i is a leaf of T_j.

Proof

Assume, for example, that s_1 is not a leaf of T_2, then there exists a vertex x such that $T_2[x,s]=T_2[x,s_1]T_2[s_1,s]$. Since, $T_1[x,s]=T_1[x,s_1](s_1,s)$, $T_1[x,s]\cap T_2[x,s]\supseteq\{s_1\}$, i.e., the two paths are not disjoint, contrary to the hypothesis. $\sqcap\!\!\!\!\sqcup$

5. Illegal subgraphs in an MCE.

This section deals with an MCE for which $deg(s) = 3$ and shows several subgraphs which cannot be contained in an MCE. The proofs of these lemmas are very similar to the proofs of Lemmas 4.1 and 4.2, so we will be more brief.

Lemma 5.1

Let F_1 be the graph of Fig 5.1. If G contains the graph F_1 as a subgraph and $deg(v;G) = 3$ then for $e_2 \neq (s,p)$, (G,s,e_1,e_2) is not an MCE.

Remark: The vertices s, p and q are connected to $G-F_1$; (p,q) may also belong to G.

Proof

If G is an MCE then by Lemma 4.3 $deg(s;G)=3$. Let $\Gamma_E(s;G)=\{e_1,e_2,e_3\}$, then since by definition $e_1=(s,v)$ and $e_2 \neq (s,p)$, we have $e_3=(s,p)$. By the minimality of G, $H=h(G-b)$ is not an MCE. We first show that $\kappa(H)=3$, then use the three spanning trees of H to construct trees for G.

Claim: $\kappa(H) = 3$.

Proof of Claim: Obviously, $\Gamma(s;G) = \{u,p,v\}$ is a separating set of G. Applying Lemma 2.1 to $\Gamma(s;G)$ shows that except perhaps for $x=v$ or $y=p$ $\kappa(x,y;G-b)=3$. We need not prove the connectivity

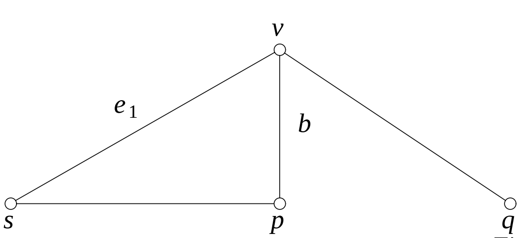


Figure 5.1 F_1

for v since $deg(v;G-b)=2$ and thus it does not belong to H. If $deg(p;G)=3$ then p does not belong to H. Thus, for all $x,y \in V(H)$, $\kappa(x,y;G-b)=3$, from which the claim follows. Otherwise, $deg(p;H)\geq 3$; thus by Lemma 2.2 $\kappa(H)=3$. (For both cases we have equality since $deg(s;H)=3$.)

Define e_2^H to be the sole member of the set $\Gamma_E(s;H)-\{(s,q),e_3\}$. Since G is an MCE, $(H,s,(s,q),e_2^H)$ satisfies E3TP. Let T_i^H $(i=1,2,3)$ be the corresponding trees and $\prec^H$ the $s-t$ ordering. Note that $s \prec^H q$ and $q \prec^H w$ for all $w \in V(H)-\{s,q\}$.

Case 1: $deg(p;G)>3$.

In this case $p \in H$, so $e_3^H = (s,p)$. The $s-t$ ordering for G is defined as $s \prec v \prec q$ and agrees with $\prec^H$ on all other vertices. The trees are

$$T_1 = T_1^H-(s,q)+(s,v)+(v,q),$$

$$T_2 = T_2^H+(v,q),$$

$$T_3 = T_3^H+(p,v).$$

Case 2: $deg(p;G) = 3$.

The $s-t$ ordering for G is defined as $s \prec v \prec p \prec q$ and agrees with $\prec^H$ on all other vertices. Let $\Gamma(p) = \{s,v,p_1\}$. Then by definition, $e_3^H = (s,p_1)G$. Define

$$T_1 = T_1^H-(s,q)+(s,v)+(v,q)+(v,p),$$

$$T_2 = T_2^H+(v,q)+(p_1,p),$$

$$T_3 = T_3^H-(s,p_1)+(s,p)+(p,p_1)+(p,v).$$

□

Lemma 5.2

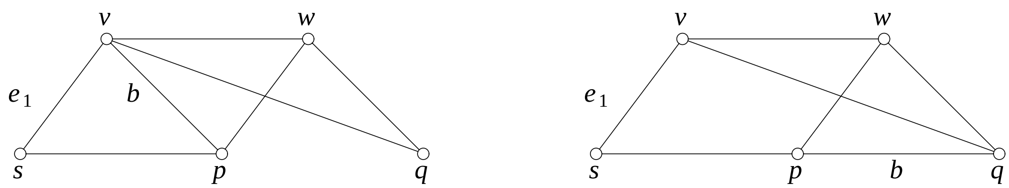
Let F_2 and F_3 be the graphs of Fig 5.2. If G contains either F_2 or F_3 and v,w have no additional neighbors, then for $e_2 \neq (s,p)$ (G,s,e_1,e_2) is not an MCE.

Proof

$G-b$ is 3-connected. Thus G cannot be an MCE. □

Lemma 5.3

Let F_4 be the graph of Fig 5.3. If G contains F_4, $e_2 \neq (s,p)$ and v,w have no additional neigh-



F_2 F_3

Figure 5.2

bors, then (G,s,e_1,e_2) is not an MCE.

Remark: W.l.g., $(p,q)G$, since otherwise we have the graph F_3.

Proof

It is easy to show that $H = h(G-b)$ is 3-connected. Thus, if (G,s,e_1,e_2) is an MCE, $(H,s,(s,q),e_2)$ satisfies E3TP. Let T_i^H $(i=1,2,3)$ be the corresponding trees and $\prec^H$ the $s-t$ ordering. By Lemma 4.4 p is a leaf of T_1^H, T_2^H and q is a leaf of T_2^H, T_3^H. W.l.g., $T_1^H[p,s] = (p,q,s)$. As $(p,q) \in T_1^H$ then

$T_1 = T_1^H - (q,s) - (p,q) + (q,v) + (w,v) + (v,s) + (p,w)$,

and $T_2 = T_2^H + (v,q) + (w,q)$,

W.l.g, $T_3^H[q,s] = (q,p,s)$ then let

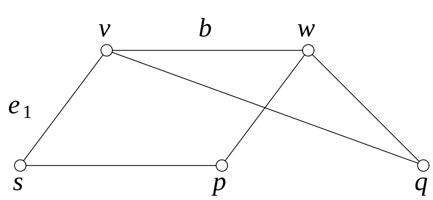
$T_3 = T_3^H - (q,p) + (q,w) + (w,p) + (v,w)$

$\prec$ is defined as the extension of $\prec^H$ to G which satisfies $s \prec v \prec w \prec \min\{z \mid z \in V(G) - s$ ordered by $\prec^H\}$. $\square$

We now come to the main lemma of this section.

Lemma 5.4

Let (G,s,e_1,e_2) be an MCE such that $\{s,p,q\}$ is a separation set with $(s,p) \in G$ and



F_4

Figure 5.3

$e_1,e_2 \neq (s,p)$. Then each component of $G-\{s,p,q\}$ contains more than two vertices.

Proof

Let B be a component of $G-S$. If $|V(B)|=1$ then the subgraph spanned by S and B is the graph F_1. If $|V(B)|=2$ then the subgraph spanned by S and B is one of the graphs F_i $i=2,3,4$. In both cases G is not an MCE. $\sqcap\!\!\!\!\sqcup$

6. E3TP and 3-connected graphs.

We give now the main result of this paper which implies Theorem 1.2.

Theorem 6.1

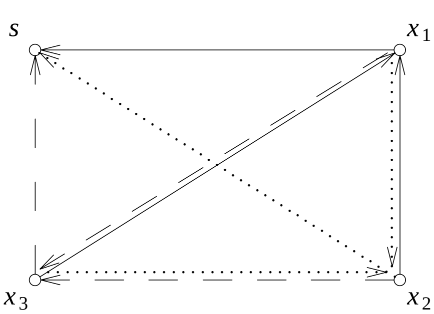
Let G be a 3-connected graph, $s \in V(G)$ and $e_1,e_2 \in \Gamma_E(s)$, then (G,s,e_1,e_2) satisfies E3TP.

Proof

As shown in Fig 6.1 if $|V(G)|=4$, E3TP is satisfied. Thus, $|V(G)|>4$. Assume the theorem is false and let (G,s,e_1,e_2) be an MCE. By Lemma 4.3, $deg(s)=3$. Let $\Gamma_E(s)=\{e_1,e_2,e_3\}$, where $e_3=(s,p)$. By Lemma 4.1, $\kappa(G/e_3)=2$. Therefore, there is a vertex q such that $S=\{s,p,q\}$ is a separating set. Since s must have an edge in each component, $G-S$ contains only two components G^1,G^2 and since $deg(s;G)=3$ and $(s,p) \in G$, $(s,q)E(G)$. By Lemma 5.4 $|V(G^i)|>2$ $(i=1,2)$.

Case 1: $(p,q)E(G)$.

As there are three disjoint paths from q to s we may assume, w.l.g., that at least two of them pass through G^2+s+p. Since $deg(s)=3$ one of these paths contains e_3. Also, there is always at least one path



$s \{x_1 \{x_2 \{x_3$

Figure 6.1

in G^1. Let $G^2 = G/G^1$ (the contraction of G^1 to a single vertex u) and G^1 the graph resulting from G by replacing G^2 by F_4 (see Fig. 6.2). I.e. $V(G^1) = V(G) - V(G^2) + \{v, w\}$ and $E(G^1) = E(G) - E(G^2) + (v, w) + (s, v) + (v, q) + (w, p) + (w, q)$; ($u, v$ and w are new vertices.) Now we prove that G^1 and G^2 are 3-connected.

Claim 1: G^1 is 3-connected.

Since there is at least one path between any two vertices of s, p, q containing only vertices of G^1, any two vertices $x, y \in \{s, p, q\}$ are 3-connected. For $x \neq s, p, q$ there exists an $x - \{s, p, q\}$ fan. Thus, by Lemma 2.5, G^1 is 3-connected.

Claim 2: G^2 is 3-connected.

As it can be verified that the connectivity between any two vertices $\neq q$ is 3, the claim follows by Lemma 2.2.

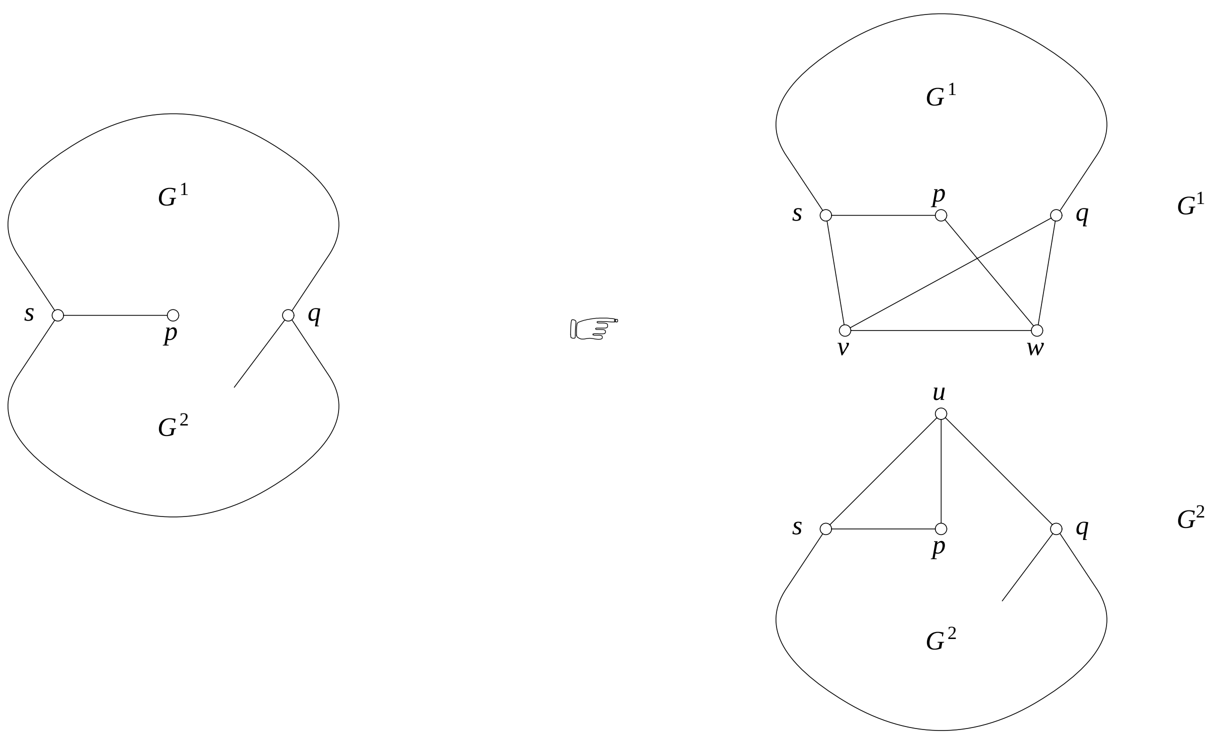


Figure 6.2

Case 1.1 $t \in G^2$:

As $|V(G^i)|>2$ $(i=1,2)$, $|V(G^i)|<|V(G)|$ therefore $(G^1,s,e_1,(s,v))$ and $(G^2,s,(s,u),e_2)$ satisfy E3TP. (Note that in G^1 v plays the role of t.) Let T_1^1,T_2^1,T_3^1 be the trees in G^1, $\prec^1$ an $s-v$ ordering and T_1^2,T_2^2,T_3^2 the trees in G^2 with an $s-t$ ordering $\prec^2$. For $i=1$, 2, 3, define: $T_i = T_i^1 \cup T_i^2 - \{u,v,w\}$. (Note that (s,p) belongs to both T_3^1 and T_3^2.)

Assume $s \prec^1 x_1 \prec^1,...,x_{l_1} \prec^1 q$ are the vertices of $G^1 \cup S$ and $s=y_1 \prec^2, \ldots, \prec^2 y_{l_2}=t$ are the vertices of G^2. Define $\prec$ to agree with $\prec^1$ on vertices of $G^1 \cup S$ and with $\prec^2$ on vertices of G^2 also if $x \in G^1 \cup S$ and $y \in G^2$ then $x \prec y$. We prove now that T_1,T_2,T_3 satisfy E3TP.

(E1) It is easy to verify that for each vertex x there exists a $x-s$ path on each T_i. Thus each of the T_i's is connected and span G.

To show that T_i contains no cycles we check the number of edges of T_i. For example: $|E(T_1)| = |E(T_1^1)|-2+|E(T_1^2)|-3 = |V(G^1)|-1+|V(G^2)|-1-5 = |V(G^1)|+|V(G^2)|-7 = |V(G)|+6-7 = |V(G)|-1$. Thus, T_1 is a spanning tree.

(E2) If $x=p$ then $T_3[p,s]=(p,s)$ $(= T_3^1[p,s] = T_3^2[p,s])$, which is disjoint of $T_i[p,s]$ $(i=1,2)$.

If $x \in G^1$.

$$T_1[x,s] = T_1^1[x,s], \tag{1}$$

$$T_2[x,s] = T_2^1[x,q]T_2^2[q,s] \text{ and} \tag{2}$$

$$T_3[x,s] = T_3^1[x,s] = T_3^1[x,p](p,s). \tag{3}$$

By E3TP in G^1, $T_3[x,s]$ is disjoint of $T_1[x,s]$ and of $T_2^1[x,q]$. It remains to show that $T_3[x,s]$ is disjoint of $T_2^2[q,s]$. This follows because the former lies entirely in G^1+s and the latter in G^2+q+s.

The remaining cases ($x \in G^2$ and $x=q$) are similar.

(E3) By the construction, only edges belonging to the original graph G appear in T_i. Therefore, for each tree T_i $(i=1,2)$ $\Gamma_E(s;G) \cap T_i = \{e_i\}$.

(E4) To prove that $\prec$ is a $s-t$ ordering we first show condition (S1), i.e. that s is the minimal vertex of $\prec$. The maximal vertex of $\prec$ is equal to the maximal vertex of $\prec^2$. Note that by E3TP in G^2, $\prec^2$ does not obtain that maximum on q or p. By Lemma 3.1 it remains to show that (T_1,T_2) are ordered by $\prec$. We show now that $T_1[x,s]$ is $\prec$-decreasing for $x \in G$. Assume $x \in G^1$, then $T_1[x,s]=T_1{}^1[x,s]$. As $\prec$ and $\prec^1$ agree on G^1, $T_1[x,s]$ is $\prec$-decreasing. If $x \in G^2$ then $T_1[x,s]=T_1{}^2[x,q]T_1{}^1[q,s]$. By E3TP on G^2 $T_1{}^2[x,q]$ is $\prec^2$-decreasing and as before $T_1{}^1[q,s]$ is $\prec^1$-decreasing. Now $\prec$ agrees with $\prec^1$ on G^1 and with $\prec^2$ on G^2. Therefore, $T_1[x,s]$ is $\prec$-decreasing. The proof that $T_2[x,s]-s$ is $\prec$-increasing is similar.

Case 1.2 $t \in G^1$:

As before, $(G^1,s,e_2,(s,v))$ and $(G^2,s,(s,u),e_1)$ satisfy E3TP. We proceed as in case 1.1 to obtain three trees T_1, T_2, $T_3 \subseteq G$ satisfying (E1-E4). In particular, there exists an $s-v$ order $\prec$ ordering (T_1,T_2) . Since $\Gamma(s;T_1)=\{t\}$, we may use Lemma 3.2 to obtain an $s-t$ order $\prec'$ such that (T_2,T_1) is ordered by $\prec'$. Thus (T_2,T_1,T_3) are three trees which demonstrate that (G,s,e_1,e_2) satisfies E3TP.

Case 2: $(p,q) \in E(G)$

In this case we construct $G^1 = G/G^2$ and $G^2 = G/G^1$. The rest of the proof is identical to case 1. $\sqcap\!\!\!\!\sqcup$

7. Conclusions.

The results of this paper and [IR] suggest the following conjecture:

Conjecture

If a graph G is k-connected and $s \in G$ then there exist k spanning trees of G such that the paths from any vertex v to s on the spanning trees are disjoint.

In this paper we used $s-t$ ordering to ensure that the paths $T_1[v,s]$ and $T_2[v,s]$ are disjoint. We have not been able to extend this method to $k>3$. Also, since our proof is much more complicated that

that of $k=2$, new ideas are required.

A similar conjecture can be stated for edges: Namely, if the graph is k-edge connected then for every vertex s there exist k spanning trees such that the paths from any vertex v to s on the spanning trees are edge-disjoint.

Both conjectures were proven for $k=2$ [IR]. It seems that methods similar to those of this paper are sufficient to prove the edge-conjecture for $k=3$. However, we feel that it is more promising to show that for all k the vertex-conjecture implies the edge-conjecture (or vice versa).

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