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ON STABILIZATION OF DISCRETE-EVENT PROCESSES

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Abstract

Discrete-Event processes are modeled by state-machines in the Ramadge-Wonham framework with control by a feedback event-disablement mechanism. In this paper concepts of stabilization of discrete-event processes are defined and investigated. We examine the possibility of driving a process (under control) from arbitrary initial states to a prescribed subset of the state set and then keeping it there indefinitely. This stabilization property is studied also with respect to 'open-loop' processes (i.e., uncontrolled processes) and their asymptotic behavior is characterized. To this end, such well known classical concepts of dynamics as invariant-sets and attractors are redefined and characterized in the discrete-event control framework. Finally, we provide polynomial time algorithms for verifying various types of attraction and for the synthesis of attractors.

1. Introduction

This paper is a preliminary investigation of the concepts of stabilization of discrete-event processes (DEP). We adopt a slightly modified version of the framework proposed by Ramadge and Wonham [1-3] for the study of DEP. Our model is thus a state machine with a means of external control: a feedback event-disablement mechanism. Unlike [4-6], we consider a state model describing the possible order of elementary events but not their exact timing.

In most of the works concerning supervisory control of DEP (e.g., [7-10]) it is assumed that the initial state of the process is fixed, known apriori and one of the 'legal' states of the process. The control problem is then to synthesize a supervisor which confines the behavior of the process, initialized at the prespecified initial state, to within legal bounds. However, there are cases in which either the initial state is not one of the legal states of the process or it is unknown apriori. In such cases the question of stabilization is of great interest.

In this paper we study the ability of a process to reach a set of target states from an arbitrary initial state and then remain there indefinitely. This stabilization property is examined under different control strategies. To this end, the classical concept of attraction [11] is reformulated and characterized in our framework. Polynomial time algorithms are provided for the verification of different types of attraction.

This paper is organized as follows. In the remainder of this section we give some terminology and notation. Invariant sets of states and realizable processes are defined in section 2. In section 3, the notion of strong attraction is introduced and examined with respect to processes without external control. Further, an efficient algorithm for computing the asymptotic behavior of such processes is proposed. Section 4 develops control strategies under which strong attraction can be achieved. To this end, a weaker form of attraction is introduced. An efficient algorithm for computing the region of weak attraction is provided in section 5. In section 6, an illustrative example is given and the relation between attraction and recovery of failure is mentioned.

1.1 Processes

Let Σ be a finite alphabet (event set). A *process* over Σ is modeled as a finite (directed) graph $G=(V,E)$ where V is a set of states (vertices) and $E \subseteq V\times\Sigma\times V$ is a set of edges. An edge of G is thus an ordered triple $e=(v,\sigma,u)\in E$ and it is said to be *directed* from v to u. The state v is called the *start-state* of e, the state u is called the *end-state* of e and $\sigma\in\Sigma$ is the event associated with e. If $(v,\sigma,u)\in E$ we say that v is a *predecessor* of u and u is a successor of v. Edges with the same start state and the same end state are called *parallel*. It is assumed that there are no two edges going out of the same state associated with the same event, that is, for each pair of edges in E

$$[\ (v,\alpha,u),(v,\beta,w)\in E \quad \text{and} \quad \alpha=\beta\ ] \qquad \text{implies} \qquad u=w\ .$$

We interpret G as a device that starts its execution at an arbitrary state $v\in V$ (v may be determined by a nondeterministic mechanism in G or forced externally) and thereafter executes a sequence of state transitions as permitted by E.

A *path* is a finite sequence of edges $e_1,e_2,\ldots,e_n$ such that the end state of e_{i-1} is the start state of e_i. The number of edges in a path is called the length of a path. The start of the path is the start state of e_1 and its end is the end state of e_n. To each path $(v_0,\sigma_1,v_1),(v_1,\sigma_2,v_2),\ldots,(v_{n-1},\sigma_n,v_n)$ there corresponds a unique (state) trajectory $v_0,v_1,\ldots,v_n$. Further, if $v_0=v_n$ the trajectory is said to be closed. A closed trajectory in which no state (except the start and end states) appears more than once is called a *cycle*. A graph without cycles is called *acyclic*.

A state v is *reachable* from a state u if there exists a path from u to v. A state v is said to be reachable from a subset of states A if v is reachable from at least one state in A. The *reach* of A in G, denoted $r_G(A)$, is defined as the set of all states in G that are reachable from A.

Let $G=(V,E)$, $\varnothing\neq A\subseteq V$. We say that a state $v\in V-A$ is connected to A if there is a path from v to a state in A. Further, G is called *A-connected* if each $v\in V-A$ is connected to A. A process $G'=(V',E')$ is called a *subprocess* of the process $G=(V,E)$, denoted $G'\subseteq G$, if $V'\subseteq V$ and $E'\subseteq E$.

The *union* of two processes $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ is another process G_3 (written $G_3=G_1\cup G_2$) whose state set is $V_3=V_1\cup V_2$ and whose edge set is $E_3=E_1\cup E_2$. If v is a state in G then $G-v$ denotes the subprocess of G obtained by deleting v from G. Deletion of a state always implies the deletion of all edges incident on that state. If e is an edge in G, then $G-e$ is a subprocess of G obtained by deleting e from G. Deletion of an edge does not imply deletion of its end states.

A subprocess $G'=(V',E')\subseteq G=(V,E)$ is called an *induced* subprocess if E' contains all the edges of E whose end points are in V'; in this case we say that G' is induced by V'. The process induced by V' is denoted $\langle V'\rangle_G$.

1.2 Supervisors

As in [1], we assume that Σ consists of two disjoint subsets Σ_u and Σ_c: uncontrolled and controlled events. Events in Σ_c can be disabled by external control while events in Σ_u cannot be prevented from occurring. Clearly, this classification of Σ induces a similar classification of E, that is, $E=E_u\cup E_c$ where $E_u=E\cap(V\times\Sigma_u\times V)$ and $E_c=E-E_u$.

A *supervisor* for G is a map $S: V\rightarrow 2^{\Sigma_c}$. For each state $v\in V$ the supervisor specifies a subset of controlled events that must be disabled.

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The concurrent operation of the process G and a supervisor S, denoted (S/G) and called the closed-loop process, is defined as the subprocess (V, E^S) of (V, E) satisfying the condition that for all $e = (v, \sigma, u) \in E$

$$e \in E^S \quad \text{iff} \quad \sigma \notin S(v) \ .$$

2. Invariant Sets of States and Realizable Processes

Let $G = (V, E)$ be a process and let $A \subseteq V$, $E' \subseteq E$. We say that A is *E'-invariant* iff

$$(\forall (v, \sigma, u) \in E') \qquad v \in A \Rightarrow u \in A \ .$$

That is, there is no edge in E' leading out of A. We remark that the important special case where A is E_u-invariant has been discussed in [2], in connection with a modular-approach solution for the problem of maintaining a predicate on V invariant.

A subprocess $G' = (V', E')$ of the process $G = (V, E)$ is called *realizable* iff

$$(\forall (v, \sigma, u) \in E_u) \qquad v \in V' \Rightarrow (v, \sigma, u) \in E' \ .$$

That is, a subprocess $G' \subseteq G$ is realizable iff every uncontrolled edge going out of a state in G' is an edge of G'. Moreover, it is easily seen that a subprocess $G' = (V', E')$ is realizable iff there exists a supervisor S such that the closed-loop process (S/G) and the subprocess G' have the same 'behavior' in the sense that for each state $v \in V'$, the set of all paths starting at v is the same in G' and (S/G). In fact, the notion of a realizable subprocess is closely related to the concept of controllable language [1].

3. Strong Attraction

In this section we examine some properties of 'open-loop' processes, i.e., processes without external control. First we introduce the concept of strong attraction.

3.1 Strong attractors

Let $G = (V, E)$ be a process and let $A, B \subseteq V$ such that $\varnothing \neq A \subseteq B$. We say that A is a *strong attractor* for B w.r.t. G, denoted $A \overset{G}{\Leftarrow} B$, iff the following conditions are satisfied:

(a1) A is E-invariant.

(a2) for each state $v \in r_G(B)$ there is a path that starts at v and ends in A.

(a3) there are no cycles of G in $r_G(B) - A$.

Thus, if $A \overset{G}{\Leftarrow} B$ then whenever the process G is initialized at state $v \in B$ it always reaches A within a finite number of state transitions and remains in A.

We show now that for each nonempty E-invariant subset A of V there exists a unique largest subset for which A is a strong attractor. To this end, let $\varnothing \neq A \subseteq V$ be E-invariant, and define $T_G(A)$ to be the (finite) class of all subsets of V for which A is a strong attractor, that is,

$$T_G(A) = \{ B \subseteq V \mid A \subseteq B \quad \text{and} \quad A \overset{G}{\Leftarrow} B \} \ .$$

Proposition 3.1

The class $T_G(A)$ is nonempty and closed under set union.

Most of the proofs of the Propositions that appear in this paper will be omitted for the limitation of space.

An immediate consequence of proposition 3.1 is that $T_G(A)$ has a unique maximal element. The maximal set B for which $A \overset{G}{\Leftarrow} B$ is denoted $\Lambda_G(A)$ and called the *region of strong attraction* of A w.r.t. G.

For a subset A which is not E-invariant we will say that $\Lambda_G(A) = \varnothing$. If $\Lambda_G(A) = V$ we say that A is a *global* strong attractor w.r.t. G (denoted $A \overset{G}{\Leftarrow}$). In cases of no confusion we shall not mention the underlying process and write, e.g., that A is a global strong attractor. It is readily verified that in the case of global strong attraction conditions (a2) and (a3) can be written as

(a2') G is A-connected.

(a3') $G - A$ is acyclic.

3.2 Asymptotic behavior

The meaning of a subset $A \subseteq V$ being a global strong attractor is that there exists a number $N \leq |V - A|$ such that every trajectory of G of length greater than N ends in A. Further, the subset A is reachable from each state in V. In other words, initializing the process at an arbitrary state $v \in V$ causes the process to reach a state in A in a finite number of state transitions. Once the process reaches a state in A it remains in A.

A natural question that arises is whether we can maximally restrict the state domain in which the process, initialized at an arbitrary state, can be 'found' after a sufficient large number (bounded by $|V|$) of state transitions. That is, we are interested in the asymptotic 'behavior' of the process.

Thus, let $G = (V, E)$ and let $g(G)$ be the (finite) class of all subsets of V that are global strong attractors w.r.t. G. That is,

$$g(G) = \{ A \subseteq V \mid A \overset{G}{\Leftarrow} \} \ .$$

First we need the following obvious observations.

Observation 3.2

The state set of G is a global strong attractor (w.r.t. G).

Observation 3.3

Let C be a cycle of G. Then C is a cycle of $<A>_G$ for every $A \in g(G)$.

Using observations 3.2 and 3.3, the following proposition is readily proved.

Proposition 3.4

Let $G = (V, E)$. Then $g(G) \neq \varnothing$, and if $A_1, A_2 \in g(G)$ then

$$\varnothing \neq A_1 \cap A_2 \in g(G) \ .$$

Proposition 3.4 implies that the finite class $g(G)$ contains a unique infimal element w.r.t. inclusion, which is denoted $\mathit{inf}[g(G)]$. Further, this infimal element satisfies the condition that

$$\mathit{inf}[g(G)] = \cap \{ A \mid A \in g(G) \} \ .$$

For an effective computation (i.e., a polynomial time algorithm) of the minimal global strong attractor we need the following proposition.

Proposition 3.5

Let $G = (V, E)$ and $v \in V$. Then $v \in \mathit{inf}[g(G)]$ if and only if either

(i) v is reachable from a state of a cycle in G ; or

(ii) v has no successors.

Proof

For abbreviation let $W \triangleq \mathit{inf}[g(G)]$.

(If). Clearly, every global strong attractor of G contains all the states in G which are 'dead-end', namely without outgoing edges. Otherwise, condition (a2') cannot be satisfied. Thus condition (ii) is a sufficient one. As regards condition (i), we note that conditions (a1) and (a3') imply that every cycle of G is contained in every global strong attractor. Moreover, since every global strong attractor A is E-invariant it follows that every state reachable from a state in A must be also in A. So we conclude that every state reachable from a state of a cycle in G is in W, which is one of the global strong attractors of G.

(Only if). Fix $v_o \in W$ and suppose, towards a contradiction, that v_o does not satisfy conditions (i) and (ii), that is,

(iii) v_o has at least one successor ; and

(iv) the subset X of all states in V from which v_o is reachable satisfies the condition that every state in X is not a state of a cycle in G.

We shall show now that W–X is a global strong attractor, contradicting our assumption that W is the minimal one.

Let Y be the set of all states in W from which v_o is reachable, i.e., $Y = W \cap X$. Clearly, Y is not empty (since $v_o \in W \cap X = Y$) and $W - X = W - Y$. First we claim that $W - Y$ is E-invariant. To see this, suppose $W - Y$ is not E-invariant, and that for some $u \in W - Y$ there exists an edge $(u, \sigma, w) \in E$ such that $w \notin W - Y$. By the definition of W it is clear that $W \overset{G}{\Leftarrow}$ and thus the E-invariance of W implies $w \in W$. Since

$$(\; w \in W \;\text{ and }\; w \notin W - Y \;) \quad \text{implies} \quad w \in Y \subseteq X \,,$$

it follows that v_o is reachable from w. Consequently, $(u, \sigma, w) \in E$ implies that v_o is reachable also from u, i.e., $u \in Y$, contradicting our assumption that $u \in W - Y$. So $W - Y$ is E-invariant.

Next we have to show that G is $(W - Y)$-connected, that is, there exists a path from each state in $V - (W - Y)$ to a state in $(W - Y)$. First we consider the state v_o. Since $v_o \in Y \subseteq W$ and W is E-invariant then every successor v_1 of v_o is in W (by assumption (iii), v_o has at least one successor). Further, $v_1 \notin Y \subseteq X$ since otherwise v_o is reachable from v_1, meaning that v_o is a state of a cycle, in contradiction to assumption (iv). Thus, $v_1 \in W - Y$ and v_o is connected to $(W - Y)$. Moreover, by the definition of X, v_o is reachable from every state in X and thus each state in X is connected to $(W - Y)$. Finally, W is a global strong attractor and thus, by condition (a2'), every state in $V - W$ is connected either to $(W - Y)$ or to $Y \subseteq X$, which is connected to $W - Y$.

It remains to be shown that $G - (W - Y)$ is acyclic, namely that every cycle of G intersects $(W - Y)$. To this end, let C be a cycle of G. Since $G - W$ is acyclic and W is E-invariant then C is contained in W. Further, by assumption (iv), every state in $Y \subseteq X$ is not a state of C and thus C must be contained in $(W - Y)$. That is, $G - (W - Y)$ is acyclic.

To summarize, we have showed that $W - Y$ is also a global strong attractor w.r.t. G, contradicting our assumption that $W = \mathit{inf}[g(G)]$.

□

Using proposition 3.3 and the *transitive closure* of G (i.e., the directed graph in which there is an edge from v to u iff there is a nonempty path from v to u in G [12, Ch. 1]), the infimal global strong attractor $\mathit{inf}[g(G)]$ can be computed in polynomial time.

4. Weak Attraction

In this section we introduce a weaker form of attraction which can be obtained under a suitable control.

4.1 Weak attractors

Let $G = (V, E)$, $\varnothing \neq A \subseteq B \subseteq V$. The subset A is called a *weak attractor* for B w.r.t. G, denoted $A \overset{G}{\leftarrow} B$, iff there exists a supervisor S such that $A \overset{(S/G)}{\Leftarrow} B$.

Clearly, strong attraction implies weak attraction but the converse is in general not true. Further, it is easily seen that a necessary condition for a subset A to be a weak attractor for another subset is that A be E_u-invariant.

Necessary and sufficient conditions for an E_u-invariant subset A to be a weak attractor for B are given by the following proposition.

Proposition 4.1

Let $G = (V, E)$, $\varnothing \neq A \subseteq B \subseteq V$, such that A is E_u-invariant. Then $A \overset{G}{\leftarrow} B$ if and only if there exists a subprocess $G' = (V', E')$ of G such that $B \subseteq V'$ and the following conditions are satisfied:

(b1) G' is A-connected.

(b2) G' is realizable.

(b3) G'–A is acyclic.

Corollary 4.1

If A is E_u-invariant and $G' = (V', E')$ satisfies condition (b1)–(b3) then $A \overset{G}{\leftarrow} V'$.

Proposition 4.1 provides necessary and sufficient conditions for the solvability of the Weak Attraction Problem (WAP), namely given a process $G = (V, E)$ and subsets $A \subseteq B \subseteq V$, verify whether WAP is solvable or not. Notice that if $\Sigma_u = \varnothing$ (i.e., every edge of G is controlled) then WAP is solvable iff each state in B is connected to a state in A. However, if the former condition does not hold (i.e., $\Sigma_u \neq \varnothing$) then WAP is not necessarily solvable even if the latter condition is satisfied.

So far we considered only E_u-invariant subsets of V as candidates for weak attractors. Clearly, this is a necessary condition. Suppose, however, that we are given two subsets A and B, such that $A \subseteq B \subseteq V$ and A is not E_u-invariant. An interesting question is whether there exists a subset $A' \subseteq A$ such that $A' \overset{G}{\leftarrow} B$. That is, find (if exists) a subset A' of A for which a supervisor S can be synthesized, so that from each initial state $v \in B$, the closed-loop process (S/G) reaches A' in finite number of state transitions and remains in A'.

The following intuitive proposition states that the problem above is solvable iff the maximal E_u-invariant subset of A is a weak attractor for B. The fact that every subset $A \subseteq V$ contains a unique maximal E_u-invariant subset, denoted $A^{\#}$, can be easily verified (cf. [2, sec. 7]).

Proposition 4.2

Let $G = (V, E)$, $\varnothing \neq A \subseteq B \subseteq V$. There exists a subset $A' \subseteq A$ such that $A' \overset{G}{\leftarrow} B$ if and only if $A^{\#} \overset{G}{\leftarrow} B$.

An effective computation of $A^{\#}$ is provided in [2, sec. 7], based on a fixed point characterization of $A^{\#}$. The verification whether $A^{\#}$ is a weak attractor for B can be accomplished by using the algorithm presented in section 5.

4.2 Region of weak attraction

Let $G = (V, E)$ be a process. In a previous section we showed that for every E-invariant subset A there is a (unique) maximal subset B for which $A \overset{G}{\Leftarrow} B$, and thus the notion of the region of strong attraction is well defined. In this section we examine whether an analogous notion can be defined for weak attraction. That is, given a nonempty subset $A \subseteq V$, we want to know whether the class of subsets that are weakly attracted by A is closed under set union, and hence has a maximal element.

Let A be E_u-invariant and define the class of subsets $W_G(A)$ according to

$$W_G(A) = \{ \; B \subseteq V \mid A \subseteq B \;\text{ and }\; A \overset{G}{\leftarrow} B \; \} .$$

Proposition 4.3:

Let A be an E_u-invariant subset of V. Then the class $W_G(A)$ is nonempty and closed under set union.

Since $W_G(A)$ is finite and closed under set union it follows that $W_G(A)$ contains a unique supremal element w.r.t. inclusion, denoted $\Omega_G(A)$ and called the *region of weak attraction* of A w.r.t. G. If A is not E_u-invariant we say that $\Omega_G(A)=\varnothing$. Further, if $\Omega_G(A)=V$ we say that A is a *global weak attractor* w.r.t. G, denoted $A \overset{G}{\leftarrow}$.

It is easily seen that

$$\Lambda_G(A) \subseteq \Omega_G(A)$$

for every $A \subseteq V$.

5. Computation of $\Omega_G(A)$

Fix $G=(V,E)$, $\varnothing \neq A \subseteq V$. In this section we propose an algorithm for the computation of the region of weak attraction $\Omega_G(A)$. A by-product of this algorithm is a subprocess of G satisfying conditions (b1)–(b3). Further, the question of whether A is a weak attractor for a subset $B \supseteq A$ is equivalent to the question of whether $B \subseteq \Omega_G(A)$. Thus, the algorithm provides a constructive method for verifying weak attraction.

Throughout this section we assume that A is E_u-invariant, for otherwise $\Omega_G(A)=\varnothing$.

We derive now an intuitive consequence of proposition 4.1 concerning the region of weak attraction. Since, by definition, $A \overset{G}{\leftarrow} \Omega_G(A)$, it follows by proposition 4.1 that there exists a subprocess $G'=(V',E')$ of G such that $\Omega_G(A) \subseteq V'$ and G' satisfies condition (b1)–(b3). Moreover, G' must satisfy the condition that $V'=\Omega_G(A)$. Otherwise the process G' would have been a contradiction to the assumption that $\Omega_G(A)$ is the largest subset for which A is a weak attractor.

We have proved:

Proposition 5.1

Let $G'=(V',E') \subseteq G$ be a subprocess such that $\Omega_G(A) \subseteq V'$. If G' satisfies (b1)–(b3), then

$$V'=\Omega_G(A) \ .$$

The subprocess $G'=(V',E')$ in proposition 5.1 is not necessarily unique. However, its state set V' is unique. The result of the algorithm below for computing $\Omega_G(A)$ is a subprocess of G that satisfies conditions (b1)-(b3) and whose state set is $\Omega_G(A)$. But first we need the following definition.

Let $G'=(V',E') \subseteq G=(V,E)$ be a process satisfying conditions (b1)–(b3), that is, G' is realizable and A-connected and $G'-A$ is acyclic. We say that a state $v \in V-V'$ is *G'-attractable* iff v is a predecessor of a state in V' and every uncontrolled edge of G leaving v ends in V', that is, $v \in V-V'$ is G'-attractable iff

(i) $(\exists (v,\sigma,u) \in E) \quad u \in V'$; and

(ii) $(\forall (v,\sigma,u) \in E_u) \quad u \in V'$.

Now we are ready for the following

ALGORITHM

Input: A process $G=(V,E)$ and a subset $A \subseteq V$.

Output: A subprocess P whose state set is $\Omega_G(A)$.

(1) Let $P_o \triangleq (U_o, D_o) = \langle A \rangle_G$, $j:=0$.

(2) If there are no P_j-attractable states in $V-U_j$ then $P=P_j$, stop.

(3) Let $v \in V-U_j$ be a P_j-attractable state. Define $P_{j+1} \triangleq (U_{j+1}, D_{j+1})$ as

$$U_{j+1}=U_j \cup \{ v \}$$

$$D_{j+1}=D_j \cup \{ (v,\sigma,u) \in E \mid u \in U_j \} \ .$$

$j:=j+1$, go to (2).

That is, the construction of a subprocess G' whose state set is $\Omega_G(A)$ is started from the subprocess P_o induced by A (step (1)). Then, in each iteration j a new subprocess P_{j+1} is constructed (step (3)) from P_j by adding a P_j-attractable state v together with every edge going from v to a state of P_j. This procedure terminates when P_j has no more attractable states (step (2)).

Since in each iteration the state set of P_j increases by one state, the number of iterations is bounded by $|V|$. Further, it is easily seen that the verification of step (2), namely that there exists a P_j-attractable state in $V-U_j$, is of complexity $O(|\Sigma||V|)$. Thus, the complexity of the algorithm above is $O(|\Sigma||V|^2)$.

The correctness of this algorithm is formally stated in the following theorem.

Theorem 5.1

Let $P=(U,D)$ be the process obtained in step (2). Then

(i) P satisfies conditions (b1)–(b3).

(ii) $U=\Omega_G(A)$.

For the proof of Theorem 5.1 we need the three following propositions. Intuitively, the first proposition states that A is a weak attractor for the state set of each process P_j. Formally, we have the following

Proposition 5.2

For every iteration j, the process P_j satisfies conditions (b1)–(b3), that is, P_j is realizable and A-connected and $P_j - A$ is acyclic.

The second proposition clarifies the role of attractable states.

Proposition 5.3

Let $P=(U,D) \subseteq G$ such that $A \subseteq U$ and P satisfies conditions (b1)–(b3). Then every P-attractable state $v \in V-U$ is in the region of weak attraction of A, i.e.,

$$v \in \Omega_G(A).$$

The final proposition required for the proof of theorem 5.1 characterizes the class of subprocesses of G whose state set is the region of weak attraction of A.

Proposition 5.4

Let $P=(U,D)\subseteq G$ such that $A\subseteq U$ and P satisfies conditions (b1)–(b3). Then

$$U=\Omega_G(A)$$

if and only if there are no P-attractable states in $V-U$.

Proof of proposition 5.4

Let $P=(U,D)\subseteq G=(V,E)$ such that $A\subseteq U$ and P satisfies conditions (b1)–(b3).

(If). Let X denote the set of all states in $V-U$ that are predecessors of a state in U, i.e.

$$X=\{\,x\in V-U \mid (\exists\,(x,\sigma,u)\in E) \quad u\in U\,\}\ ,$$

and suppose that every state in X is not P-attractable. We have to prove that $U=\Omega_G(A)$.

First notice that P satisfies conditions (b1)–(b3) and thus, by corollary 4.1, $U\subseteq\Omega_G(A)$. For the reverse inclusion we shall show first that none of the states in X can be in the region of weak attraction of A, i.e.,

$$X\cap\Omega_G(A)=\varnothing\ .$$

For this let $x_1\in X\subseteq V-U$ and suppose, towards a contradiction, that $x_1\in\Omega_G(A)$. According to proposition 4.1, if $(U\cup\{x_1\})\subseteq\Omega_G(A)$ then there exists a subprocess $G'=(V',E')$ of G such that $(U\cup\{x_1\})\subseteq V'$ and G' satisfies conditions (b1)-(b3). Since none of the states in X is P-attractable then there exists an edge $e_1=(x_1,\sigma,v_1)\in E_u$ such that $v_1\notin U$. The edge e_1 is uncontrolled and thus e_1, as well as v_1, must be included in G'. Otherwise G' could not be realizable (condition (b2)). Moreover, G' is A-connected and thus it must contain a trajectory from v_1 to a state in U (notice that every state in U is connected to A). Now, since $v_1\notin U$ there are two possibilities: either $v_1\in V-U-X$ or $v_1\in X$.

(i) If $v_1\in V-U-X$ then every trajectory of G' from v_1 to a state in U must include at least one state in X (this is because every predecessor of a state in U is in X). Let t be such a trajectory, namely a trajectory connecting v_1 to U, and let x_2 be the first state in X traversed by t. Subsequently, denote by t_1 the subtrajectory of t connecting v_1 to x_2, i.e., $t_1=v_1,\ldots,x_2$. Notice that none of the states of t_1 is a state in U (written $t_1\cap U=\varnothing$). Also, the condition $x_2\neq x_1$ must be satisfied in order that $G'-A$ will be acyclic (otherwise $G'-A$ will contain the cycle x_1,t_1).

(ii) If $v_1\in X$ then $x_2=v_1$ and t_1 is the empty trajectory.

Since t_1 is a trajectory of G' then x_2 is also a state of G'. So we conclude that

$$x_1\in V' \quad \text{implies} \quad x_2\in V'\ .$$

Following the argument of the previous paragraph we get that G' must contain a trajectory, say t_2, connecting $x_2\in X$ to $x_3\in X$, where $x_3\neq x_2$, $x_3\neq x_1$ and $x_3\in V'$.

Continuing this procedure we end up with the following conclusions regarding the process G' : x_1 is connected to x_2 by t_1, x_2 is connected to x_3 by t_2, $\cdots$, x_{n-1} is connected to x_n by t_{n-1}, x_n is connected to x_j, $1\le j\le n$, by t_n and

$$x_1,x_2,\cdots,x_n\in V'\ ,$$

where n is number of states in X and $x_i\neq x_j$, $i\neq j$.

It is readily verified that the trajectory $t_j,t_{j+1},\cdots,t_n$ forms a cycle in $G'-A$ (notice that $A\subseteq U$ and $t_i\cap U=\varnothing$, $1\le i\le n$). Thus we conclude that the assumption $x_1\in V'\subseteq\Omega_G(A)$ implies $X\subseteq V'$. However, the requirement from G' to be A-connected implies the existence of a cycle in $G'-A$, contradicting condition (b3). So

$$X\cap\Omega_G(A)=\varnothing\ . \tag{5.1}$$

As regards the rest of the states in $V-U$; since every path from a state in $V-U-X$ to a state in A must traverse at least one state in X it is clear that

$$(V-U-X)\cap\Omega_G(A)=\varnothing\ . \tag{5.2}$$

From (5.1) and (5.2) we get that $\Omega_G(A)\subseteq U$, which concludes the "if" part of this proof, i.e.,

$$U=\Omega_G(A)\ .$$

(Only if). Suppose $U=\Omega_G(A)$ and assume there exists a state $v\in V-U$ such that v is P-attractable. However, by proposition 5.3 it follows that $v\in\Omega_G(A)$, contradicting our assumption that U is the region of weak attraction of A.

□

Proof of Theorem 5.1

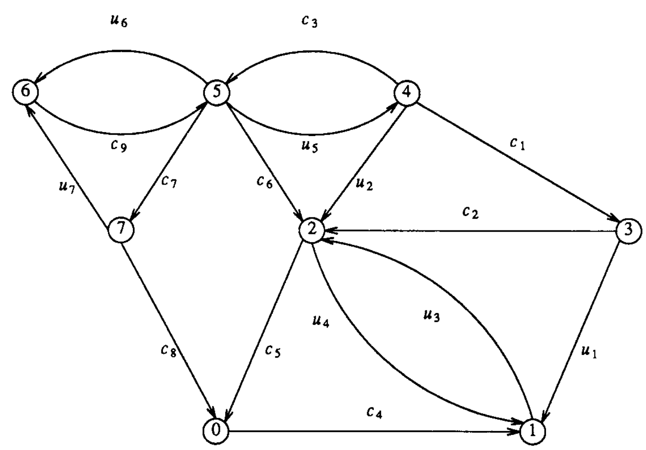
Let $P=(U,D)\subseteq G=(V,E)$ be the process obtained in step (2) of the algorithm. By proposition 5.2 it is clear that P satisfies conditions (b1)-(b3). Further, since every state in $V-U$ is not P-attractable (according to the condition of step (2)) then by proposition 5.4

$$U=\Omega_G(A)\ .$$

□

6. Example

Let $G=(V,E)$ be a process as displayed below:



The state set of G is $V=\{0,1,\ldots,7\}$, and the edge set is $E=\{u_i\}\cup\{c_i\}$. The edges denoted u_i are uncontrolled while c_i denotes a controlled edge.

First we comment that the subset $A_1=\{1,2\}$ cannot be a strong attractor for any subset of V (since A_1 is not E-invariant). Nevertheless, A_1 is E_u-invariant and thus it has a potential to become a weak attractor (e.g., by the deletion of c_5).

Next we consider the subset $A=\{0,1,2\}$. Clearly A is E-invariant, and if $B_0=A\cup\{3\}$ then A is a strong attractor for B_0 w.r.t. G. It is easily seen that B_0 is the maximal subset of V which is strongly attracted by A. That is, $\Lambda_G(A)=B_0$. Further, we remark that the region of strong attraction $\Lambda_G(A)$ can be computed in polynomial time by using the transitive closure of G (see at the end of section 3).

We examine now the weak attraction problem, namely given two subsets A, B of V, decide whether there exists a supervisor S such that A is a strong attractor for B w.r.t. (S/G). To this end, let $A = \{0,1,2\}$, $B_1 = A \cup \{3,4\}$ and $B_2 = A \cup \{7\}$. Recall that we defined weak attraction $A \overset{G}{\leftarrow} B$ as the possibility of driving G (under control) from every initial state in B to some state in A. Consequently, the deletion of the controlled edge c_3 implies $A \overset{G}{\leftarrow} B_1$. Furthermore, it is readily verified that the subprocess $< B_1 >_G$ (i.e. the subprocess induced by the states of B_1) satisfies the conditions of weak attraction, as stated in proposition 4.1.

As regards B_2, it can be shown that no subprocess of G, whose state set contains B_2, satisfies the conditions of proposition 4.1 (i.e., (b1)-(b3)). Thus we conclude that A is not a weak attractor for B_2. Intuitively, this result can be explained as follows: Suppose G is initialized at state $7 \in B_2$. Then either G reaches state 0 (and then is captured in A) or it executes u_7 and reaches state 6. Since the edge u_7 is uncontrolled (and thus cannot be removed from G) it follows that the edge c_9 must not be deleted from G. Otherwise the subset $A = \{0,1,2\}$ is not reachable from state 6. However, the latter conclusion and the fact that u_6 is uncontrolled imply the existence of the cycle $C = 6,5,6$. The cycle C prevents the guaranteed attraction of state 7 to a state in A, i.e., if G is initialized at state 7 then no control strategy can assure that G (under control) will reach the subset A after executing a finite number of state transitions.

The existence of a subprocess G' as required in proposition 4.1 can be effectively verified by using the algorithm of section 5 for computing the region of weak attraction. If we apply the algorithm to this example we obtain the following steps:

(i) Start with the subprocess $P_o = <A>_G = (\{0,1,2\}, \{u_3, u_4, c_4, c_5\})$; (step (1)).

(ii) A candidate state for the next step is any predecessor of a state in A which is P_o-attractable. Since the uncontrolled edges u_7, u_6 and u_5 lead to a state in $V-A$, none of the states 7 or 5 is P_o-attractable. Thus, choose for example state 3 and construct (step (3)) the subprocess

$$P_1 = (\{0,1,2,3\}, \{u_3, u_4, c_4, c_5, c_2, u_1\})$$

$$= <\{0,1,2,3\}>_G .$$

(iii) Only state 4 is P_1-attractable and thus construct (step (3) again) the subprocess

$$P_2 = (\{0,1,2,3,4\}, \{u_3, u_4, c_4, c_5, c_2, u_1, u_2, c_1\})$$

$$= <\{0,1,2,3,4\}>_G .$$

(iv) There are no P_2-attractable states and thus the algorithm terminates; (step (2)).

By theorem 5.1 we conclude that $\Omega_G(A) = \{0,1,2,3,4\} = B_1$, and that P_2 satisfies the conditions of proposition 4.1. Based on P_2, a control pattern achieving weak attraction of B_1 by A is readily synthesized (see the proof of proposition 4.1).

As was explained in the paragraph following proposition 5.1, the resulting process in step (2) is not unique. For example, if we had interchanged steps (ii) and (iii) we would have ended up with the process $P_2 - c_1$. Nevertheless, the region of weak attraction of A is yet B_1, since the state set of $P_2 - c_1$ is B_1. This illustrates the consequence of proposition 4.3, namely that the region of weak attraction is well defined.

Our intuitive conclusion that A is not a weak attractor for B_2 is now an immediate consequence of the fact that B_2 is not a subset of $\Omega_G(A)$.

We end this example by pointing out the close relation between attraction properties and the problem of recovery from control failures. For example, suppose $A = \{0,1,2\}$ is the 'legal' state set of V, and that a control failure may cause G to reach the illegal state 7. Since A is not a weak attractor for B_2, no control strategy can assure a guaranteed recovery (i.e., a guaranteed return of G (under control) to a legal state in A) from this control failure. On the other hand, $A \overset{G}{\leftarrow} B_1$ implies the existence of a supervisor achieving guaranteed recovery from control failures causing G to reach states 3 or 4. Such a supervisor is readily synthesized by using the output of the algorithm in section 5.

7. Conclusion

The paper has presented the concept of strong attraction which plays a key role in the investigation of the following problems. The first one is the ability of a process to reach a set of target states from an arbitrary state and then remain there indefinitely. Another problem, which is closely related to the former, is the recovery from control failures. Finally, a special kind of asymptotic behavior of a process has been characterized as its minimal strong attractor. The first two problems were examined also under control, and an efficient procedure for synthesizing controllers that improve the attraction ability of processes has been proposed. The properties of such controllers and the extension of the above results for other representations of discrete event processes are interesting topics for further research.

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